NONLINEAR RESPONSE OF COMPOSITE LAMINATES WITH RANDOM MATERIAL PROPERTIES TO RANDAM LOADING

By

Amit Kumar Onkar

DEPARTMENT OF AEROSPACE ENGINEERING

Indian Institute of Technology Kanpur

June, 2002

NONLINEAR RESPONSE OF COMPOSITE LAMINATES WITH RANDOM MATERIAL PROPERTIES TO RANDOM LOADING

A *Thesis* Submitted in Partial Fulfillment

of the Degree of

MASTER OF TECHNOLOGY

by

Amit Kumar Onkar

to the

DEPARTMENT OF AEROSPACE ENGINEERING

INDIAN INSTITUTE OF TECNOLOGY KANPUR, INDIA

June 2002

CERTIFICATE

It is certified that the work contained in the thesis entitled **NONLINEAR RESPONSE OF COMPOSITE LAMINATES WITH RANDOM MATERIAL PROPERTIES TO RANDOM LOADING**, by **Amit Kumar Onkar**, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

Dayanand Yadav

Dr. Dayanand Yadav
Professor
Dept. of Aerospace Engineering
I.I.T. kanpur 208016, India

June 2002

Synopsis

Laminated composite plates are fast replacing metal alloys in most light transport vehicles and various other industries. Increasingly, many aerospace and high-speed rail components are being fabricated with composites.

The material properties of such composite display considerable scatter because of the uncertainty involved at many levels-properties of its constituents, fabrication and manufacturing processes, geometrical parameters of laminates, fiber orientations, volume fraction, curing temperature and pressure, voids, impurities, curing time etc. This leads to variation in stiffness coefficients of the laminates and thus results in uncertainty in the response behavior. Besides this the structure may be subjected to variations in input loading. The system can best be modelled as having random material properties with random excitation. In some applications the structure undergoes large deformations, which demands the investigation of their nonlinear behavior.

The present work is an attempt to incorporate nonlinear effects in the random environment with the material properties and external loading as random. The second order statistics of static deflection, natural frequencies and forced vibration response of composite plate with random material properties and random excitation is evaluated. The basic formulation uses classical laminate theory and Von-Karman non-linear strain-displacement relationship. The system equation for the static case is obtained by using Rayleigh-Ritz method whereas the dynamic formulation uses Hamilton's principle. The numerical results for mean and SD for the responses have been presented.

ACKNOWLEDGEMENT

I take this opportunity to express my sincere gratitude to my thesis supervisor Prof. Dayanand Yadav for his constant support, encouragement and guidance throughout this work. I am highly indebt to him for all those fruitful discussions, and for having introduced me to the *Aircraft Structure* and *Random Vibration*. I really feel myself very fortunate to be associated with him. I cherish each and every moment with him.

I am very much thankful to Prof N. G. R. Iyengar for introducing me to the world of *Composite Material* and *Composite Structure* that was completely new for me. I would like to thank Prof. C. S. Upadhyay for all support and encouragement during his course work. His amicable personality always gave me pleasure to have discussion with him.

I specially thank Najir sir for his excellent library support. My sincere thanks goes to Mohite for giving all support and care during stay at IIT Kanpur.

I express my heartfelt thank to my all friends Pankaj, Ayush, Manoj, Ajit, Narmdeshwar, Seema and Shalini for their love and care during stay at IIT Kanpur. I also thank my juniors Keshri and Shadab for their love to me.

I would like to thank my labmates Seresta, Srinivas, Vaitla, Reddy, Atul, Amnaya, and Anil who always kept me fresh while working in Lab.

My feeling towards my parents and sisters Nisha and Neha can't be expressed in words but it means a lot to me that they have understood and cared more than anyone ever could. They have always been the constant source of inspiration for me.

Amit Kumar Onkar

LIST OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

1.1 GENERAL INTRODUCTION

Composites are known to have specific advantages over metals as these can be tailored to have directional properties required for the loading. This has lead to their extensive use in diverse engineering applications. Composite has a large number of parameters involved with its production and fabrication process. Change in factors such as fiber orientation, laminae thickness, volume fraction, curing temperature and pressure, voids, impurities, curing time etc. induce variations in the lamina properties. This leads to variation in stiffness coefficients of the laminates. It is practically not possible to completely control variation in all these parameters and thus dispersions in the material and geometric properties are inherent. In use, the uncertainty in the system parameters result in uncertainty in the response behavior. Further, the external loading (both static and dynamic) may also be random in nature. In many applications, like aerospace engineering, it is important for the designer to have an accurate evaluation of the structural response. For such sensitive applications, enhanced accuracy in response evaluation is possible by modeling the problem with random material properties and random external loading. Some of these structures are very often subjected to severe loadings that result in large response and consequently demand the investigation of their nonlinear behaviour.

1.2 LITERATURE SURVEY

Any structural analysis problem is characterised by the following three basic aspects: material of the structure, geometry of the structure and type of loading. In real

life problems, all the three aspects are random in nature. Significant volume of literature is available for external loading as random with material properties and geometry as deterministic. Nigam and Narayanan [1] present various classes of problems in this area.

Some published literature is available for analysis of conventional structure with random material properties. Ibrahim [2] has reviewed topics pertaining to structural dynamics with parameter uncertainties. Nakagiri et al. [3] have studied simply supported (SS) Graphite/Epoxy plates with stochastic finite element method (SFEM) taking fiber orientation, layer thickness and layer numbers as random variables and found that the overall stiffness of fiber reinforced composite (FRP) laminated plates is largely dependent on the fiber orientation. Fukunaga et al. [4] have investigated the effects of scatter in the lamina strengths, relative fiber volume fraction, and laminate stacking sequence on the ultimate strength of the hybrid laminates. Raj et al. [5] have analysed rectangular plates with and without cut-outs using higher order shear deformation theory by using combination of FEM and Monte Carlo Simulation (MCS).

Leissa and Martin [6] have analysed the vibration and buckling of rectangular composite plates, and have established that variation in fiber spacing tends to increase the buckling load by 38% and the fundamental frequency by 21%. Shinizuka and Astill [7] have employed a numerical technique to obtain statistical properties of eigen values of spring supported columns with the spring support and axial loading along with material and geometric properties as random. The method has been used to investigate the accuracy of perturbation approach for calculation of vibration and buckling modes. Salim et al. [9-12] have studied the statistical response of plates considering material properties as independent random variables (RV). The second order statistics for static deflection,

natural frequency and buckling load of rectangular plates have been studied using a first order perturbation technique (FOPT). Free vibration response has been obtained by Vaicatis [13] for beams with mass and flexural rigidity as random variables. The nonlinear forced vibration of circular and rectangular plate with various boundary conditions has been studied [15-18] by applying the Galerkin or Rayleigh Ritz methods. Singh et al. [19] has studied the nonlinear forced vibration of antisymmetric rectangular cross-ply plates by using Hamilton principle.

Chen and Soroka [20] have studied the response of a multi degree of freedom system with random properties to deterministic excitations. The system equations have been solved by perturbation technique. The second order statistics of the system response have been investigated with variation in the system property statistics. Yadav and Verma [21-22] have studied the buckling and free vibration response of thin cylindrical shell using classical laminate theory (CLT) and have employed the FOPT for obtaining the second order statistics of buckling loads and natural frequencies. Gorman [23] has presented free vibration analysis of thin rectangular plates with variable edge supports using the method of superimposition. Singh et al. [24-25] have studied the initial buckling and natural frequency of cylindrical panel and composite plate with random material properties and have obtained the second order statistics of response.

PRESENT WORK

Many researchers have studied the uncertainties in material strength and stiffness of composites. A comprehensive survey of the literature indicates that a significant volume of publication exists for response of composite beam and plates with linear strain-

displacement relation. However, the response of composite plate in random environment with nonlinear strain-displacement has not received attention to the researchers. In many applications, the deflections are large and may lead to nonlinear behaviour. The present work is an attempt to incorporate the nonlinear effects in the random environment. The second order statistics of static deflection, natural frequencies of free vibration and forced vibration response of composite plate with random material properties and random excitation is evaluated. The basic formulation uses CLT and Von-Karman non-linear strain displacement relationship. The system equation for the static case is obtained by using the Rayleigh-Ritz method whereas Hamilton's principle has been employed for the dynamic problems. Lamina material properties are modeled as basic random variables (RVs) and are assumed to be uncorrelated for the generation of the results. External excitation is assumed to be random in time. The numerical results for mean and SD for the responses for known statistics of material properties and loading are obtained. Effects of parameters like side-to-thickness ratio and aspect ratio for symmetric and anti-symmetric cross-ply laminates has been examined.

CHAPTER 2

FORMULATION

2.1 INTRODUCTION

In this chapter expressions are developed for the response statistics of composite laminated plate with random material properties having simply supported edges. Three types of responses are discussed in this study- (i) static deflection, (ii) free vibration and (iii) force vibration. As mentioned before, the classical laminate theory and Von Karman non-linear strain displacement relation have been used for the basic formulation. The system equation for the static case is developed by using Rayleigh Ritz method whereas for the dynamic cases Hamilton principle has been employed. First order Perturbation technique is used to analyse the problems and the second order response statistics has been obtain.

2.2 GENERAL FORMULATION

Consider a rectangular composite plate of constant total thickness 'h', composed of thin orthotropic layers bonded together. The origin of a Cartesian coordinate system, as shown in Figure 2.1, is located in the central plane at the left corner with x and y axes along the middle plane and the z-axis normal to this plane.

If Q_{ij} are the reduced stiffness matrix terms, the simplified relations between basic lamina properties and reduced stiffness matrix terms are given as [26]:

$Q_{11}=E_{11}/(1-\nu_{21}\,\nu_{12})$

$Q_{22}=E_{22}/(1-\nu_{21}\,\nu_{12})$

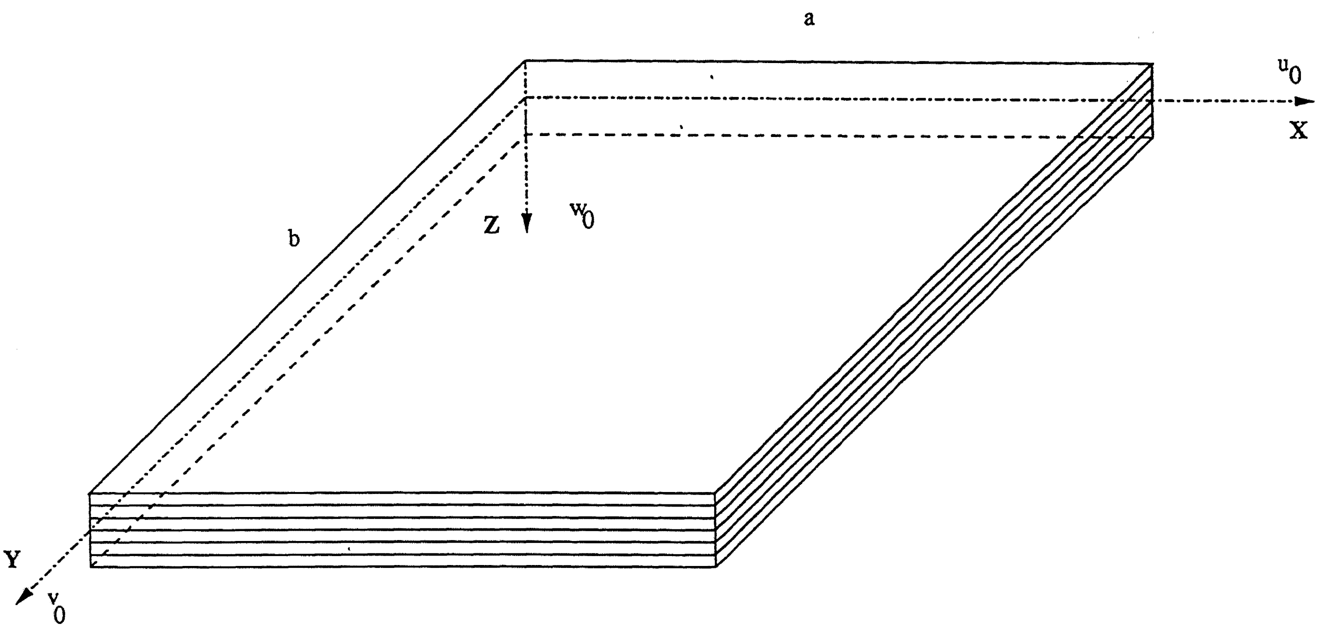


Figure 2.1- Geometry of a laminated composite plate

$$Q_{12}=\nu_{12}E_{22}/(1-\nu_{21}\nu_{12}) \tag{2.1}$$

$$Q_{21}=Q_{12} \quad \text{and} \quad Q_{66}=G_{12}$$

ν_{21} can be expressed in terms of the other material properties as:

$$\nu_{21}=\nu_{12}E_{22}/E_{11}$$

The transformed reduced stiffness matrix terms $\bar{Q}_{ij}$ are related to the reduced stiffness matrix terms Q_{ij} through the following equations [26]:

$$\bar{Q}_{11}=Q_{11}cos^4\theta+Q_{22}sin^4\theta+2(Q_{12}+2Q_{66})sin^2\theta\, cos^2\theta$$

$$\bar{Q}_{22}=Q_{11}sin^4\theta+Q_{22}cos^4\theta+2(Q_{12}+2Q_{66})sin^2\theta\, cos^2\theta$$

$$\bar{Q}_{12}=Q_{12}(cos^4\theta+sin^4\theta)+(Q_{11}+Q_{22}-4Q_{66})sin^2\theta\, cos^2\theta \tag{2.2}$$

$$\bar{Q}_{66}=Q_{66}(cos^4\theta+sin^4\theta)+(Q_{11}+Q_{22}-2Q_{12}-2Q_{66})sin^2\theta\, cos^2\theta$$

$$\bar{Q}_{16}=(Q_{11}-Q_{12}-2Q_{66})cos^3\theta\, sin\theta-(Q_{22}-Q_{12}-2Q_{66})sin^3\theta\, cos\theta$$

$$\bar{Q}_{26}=(Q_{11}-Q_{12}-2Q_{66})cos\theta\, sin^3\theta-(Q_{22}-Q_{12}-2Q_{66})sin\theta\, cos^3\theta$$

The elements of the extensional stiffness matrix A_{ij}, coupling matrix B_{ij} and bending stiffness matrix D_{ij} are given by [26]:

$$A_{ij}=\sum_{k=1}^{n}(\bar{Q}_{ij})_k(h_k-h_{k-1})$$

$$B_{ij}=(1/2)\sum_{k=1}^{n}(\bar{Q}_{ij})_k(h_k^2-h_{k-1}^2) \tag{2.3}$$

$$D_{ij}=(1/3)\sum_{k=1}^{n}(\bar{Q}_{ij})_k(h_k^3-h_{k-1}^3) \qquad i,j=1,2 \text{ and } 6$$

where 'h_k' is the thickness of the k^{th} lamina in the given laminate.

The stress and moment resultants per unit length of the plate in terms of stiffness matrices are defined as [26]:

$$\begin{bmatrix} N_i \\ M_i \end{bmatrix} = \begin{bmatrix} A_{ij} & B_{ij} \\ B_{ij} & D_{ij} \end{bmatrix} \begin{bmatrix} \varepsilon_j \\ \kappa_j \end{bmatrix} \qquad \text{(i,j=1,2,6)} \tag{2.4}$$

where ε_j and κ_j are the mid plane strains and curvature strains respectively.

The strain energy for the plate can be written as [31]:

$$U=\frac{1}{2}\int_{-h/2}^{h/2}\int_{A}(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\tau_{xy}\gamma_{xy}+\tau_{xz}\gamma_{xz}+\tau_{yz}\gamma_{yz})dA\,dz \tag{2.5}$$

The stress and moment resultants can also be defined as [31]:

$$(N_i, M_i)=\int_{-h/2}^{h/2}\sigma_i(1,z)dz \qquad (i=x,y) \tag{2.6}$$

Neglecting transverse shear effect, i.e. ($\gamma_{xz}, \gamma_{yz} = 0$) and by substituting equations (2.4) and (2.6) in (2.5) the expression for the strain energy of the plate becomes:

$$\begin{aligned} U=\frac{1}{2}\int_0^a\int_0^b\{ & A_{11}\varepsilon_x^{\,2}+2A_{12}\varepsilon_x\varepsilon_y+A_{22}\varepsilon_y^{\,2}+2A_{16}\varepsilon_x\gamma_{xy}+A_{26}\varepsilon_y\gamma_{xy}+A_{66}\gamma_{xy}^{\,2}+ \\ & 2B_{11}\kappa_x\varepsilon_x+B_{12}(\kappa_y\varepsilon_x+2\kappa_x\varepsilon_y)+2B_{22}\kappa_y\varepsilon_y+2B_{16}(\kappa_{xy}\varepsilon_x+\kappa_x\gamma_{xy})+ \\ & 2B_{26}(\kappa_{xy}\varepsilon_y+\kappa_y\gamma_{xy})+2B_{66}\gamma_{xy}\kappa_{xy}+D_{11}\varepsilon_x^{\,2}+2D_{12}\kappa_x\kappa_y+D_{22}\varepsilon_y^{\,2}+ \\ & 2D_{26}\kappa_{xy}\kappa_y+2D_{16}\kappa_{xy}\kappa_x+D_{66}\kappa_{xy}^{\,2}\}dxdy \end{aligned} \tag{2.7}$$

Assuming that the deflection may be large ..., the following Von Karman strain-displacements are used [31]:

$$\varepsilon_x=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_y=\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{2.8}$$

where ε_x, ε_y and γ_{xy} are mid plane strains and *u*, *v* and *w* are mid plane displacement in x, y and z directions.

The mid plane curvature-displacement relationship can be put as:

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \quad \kappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} \tag{2.9}$$

2.3 BOUNDARY CONDITION

The boundary condition for a cross-ply plate with all sides simply supported with edges free to move in their respective normal directions are given by the following set of equations:

Along x=0 and x=a for all y,

$$w=0:\ M_x = -D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right) - D_{12}\left(\frac{\partial^2 w}{\partial y^2}\right) = 0,$$

$$v=0:\ N_x = A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y} = 0$$

Along y=0 and y=b for all x, (2.10)

$$w=0:\ M_y = -D_{12}\left(\frac{\partial^2 w}{\partial x^2}\right) - D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right) = 0,$$

$$u=0:\ N_y = A_{12}\frac{\partial u}{\partial x} + A_{22}\frac{\partial v}{\partial y} = 0$$

2.4 STATIC DEFLECTION ANALYSIS

2.4.1 System Equations

Consider a specially orthotropic rectangular laminated composite plate subjected to a distributed sinusoidal transverse loading given by:

$$Q(x,y) = Q_0 \, Sin\frac{m\pi x}{a} \, Sin\frac{n\pi y}{b} \tag{2.11}$$

where a and b are the plate dimensions in x and y directions respectively, Q_0 is a constant giving the loading amplitude and m and n are integers. This loading form is generic and can be used to simulate different types of transverse loadings on the plate.

In case of rectangular antisymmetric crossply all the coupling elements of [A], [B] and [D] matrix identically go to zero i.e. $A_{16}=A_{26}=B_{16}=B_{26}=D_{16}=D_{26}=0$. From equation (2.7) the strain energy for the plate can be written as:

$$U=\frac{1}{2}\int_0^a\int_0^b\{A_{11}\varepsilon_x^2+2A_{12}\varepsilon_x\varepsilon_y+A_{22}\varepsilon_y^2+A_{66}\gamma_{xy}^2+2B_{11}\kappa_x\varepsilon_x+B_{12}(\kappa_y\varepsilon_x+2\kappa_x\varepsilon_y)$$

$$+2B_{22}\kappa_y\varepsilon_y+2B_{66}\gamma_{xy}\kappa_{xy}+D_{11}\varepsilon_x^2+2D_{12}\kappa_x\kappa_y+D_{22}\varepsilon_y^2+D_{66}\kappa_{xy}^2\}dxdy \quad (2.12)$$

For a sinusoidal static load defined in equation (2.11), the external work done can be represented by:

$$W=\int_0^a\int_0^b Q(x,y)\, w(x,y)\, dx\, dy \quad (2.13)$$

The following set of admissible functions can be used to satisfy the defined boundary conditions:

$$u=u_0 Cos\frac{m\pi x}{a}\, Sin\frac{n\pi y}{b}, \qquad v=v_0 Sin\frac{m\pi x}{a}\, Cos\frac{n\pi y}{b},$$

$$w=w_0 Sin\frac{m\pi x}{a}\, Sin\frac{n\pi y}{b} \quad (2.14)$$

where u_0, v_0, and w_0 are constants indicating the displacement amplitudes.

Substituting equations (2.8), (2.9) and (2.14) in equations (2.12) and (2.13) and integrating gives the total potential energy of the system as:

$$\pi=U-W \quad (2.15)$$

Minimizing the total potential with respect to u_0, v_0 and w_0 results in the following three simultaneous nonlinear algebraic equations:

$$T_1u_0+T_2v_0+T_3w_0+T_4w_0^2=0 \quad (2.16)$$

$$T_2u_0+T_5v_0+T_6w_0+T_7w_0^2=0 \quad (2.17)$$

$$T_3u_0+T_6v_0+T_8w_0+2T_4u_0w_0+2T_7v_0w_0+T_9w_0^2+T_{10}w_0^3=Q_0 \quad (2.18)$$

where the coefficients T_1, T_2,...,T_{10} depend on the plate geometry, material properties and the mode shapes. Their expressions are:

$$T_1=(m\pi/a)^2A_{11}+(n\pi/b)^2A_{66},\quad T_2=(m\pi/a)(n\pi/b)(A_{12}+A_{66})$$

$$T_3=-(m\pi/a)^3B_{11},$$

$$T_4=(-4/9mn\pi^2)S_{mn}[(m\pi/a)^3A_{11}+(m\pi/a)(n\pi/b)^2(A_{12}-A_{66})]$$

$$T_5=(m\pi/a)^2A_{66}+(n\pi/b)^2A_{22},\quad T_6=-(n\pi/b)^3B_{22} \tag{2.19}$$

$$T_7=(-4/9mn\pi^2)S_{mn}[(n\pi/b)^3A_{22}+(n\pi/b)(m\pi/a)^2(A_{12}-A_{66})]$$

$$T_8=(m\pi/a)^4D_{11}+2(m\pi/a)^2(n\pi/b)^2(D_{12}+2D_{66})+(n\pi/b)^4D_{22}$$

$$T_9=(4/3mn\pi^2)S_{mn}[(m\pi/a)^4B_{11}+(n\pi/b)^4B_{22}$$

$$T_{10}=(9/32)[(m\pi/a)^4A_{11}+(n\pi/b)^4A_{22}]+(1/16)(m\pi/a)^2(n\pi/b)^2(A_{12}+2A_{66})$$

where $S_{mn}=(1-(-1)^m)\ (1-(-1)^n)$

It is possible to obtain u_0 and v_0 in terms of w_0 from equations (2.16) and (2.17). Thereafter, substituting in equation (2.18), the resulting equation in w_0 becomes:

$$u_0=[\{(T_2T_6-T_3T_5)\,w_0+(T_2T_7-T_4T_5)\,w_0^2\}/(T_1T_5-T_2^2)] \tag{2.20}$$

$$v_0=[\{(T_2T_3-T_1T_6)\,w_0+(T_2T_4-T_1T_7)\,w_0^2\}/(T_1T_5-T_2^2)]$$

and $$L_1\,w_0+L_2\,w_0^2+L_3\,w_0^3=Q_0 \tag{2.21}$$

Where L_1, L_2 and L_3 are as follows:

$$L_1=[T_8+\{(2T_2T_3T_6-T_3^2T_5-T_6^2T_1)/(T_1T_5-T_2^2)\}]$$

$$L_2=[T_9+\{(3T_2T_3T_7+3T_2T_4T_6-3T_3T_4T_5-3T_1T_6T_7)/(T_1T_5-T_2^2)\}] \tag{2.22}$$

$$L_3=[T_{10}+\{(4T_2T_4T_7-2T_5T_4^2-T_1T_7^2)/(T_1T_5-T_2^2)\}]$$

2.4.2 Mean and variance of static response – perturbation technique

First order perturbation approach has been adopted for obtaining the deflection statistics with randomness in material properties and loading on a flat plate. Here it is

assumed that all the material properties and loading components are independent of each other. Also it is assumed that the dispersion of each random variable about its mean value is small, which is true in most engineering applications.

Any random variable may be split up as the sum of the mean variable and zero mean random part with generality as:

$$R=R^{d}+R^{r}, \tag{2.23}$$

where superscript '*d*' denotes the mean value, which is deterministic, and '*r*' denotes the superimposed zero mean random component.

Using Taylor series expansion the random part of the dependent variables can be expressed in terms of the independent variables. Consider L_i as the system operator, w as the displacement response of the system, Q to denote the force applied to the system and b_l as the basic material properties. The primary variables are assumed to be the basic material properties b_l and the elements of the applied load q_k. As assumed, the random part in the primary variables is small in magnitude compared to their mean values, the second and higher order terms are neglected and the expression may be put as:

$$L_i^r=\sum_l(\partial L_i^d/\partial b_l^d)b_l^r$$

$$w_0^r=\sum_l(\partial w_0^d/\partial b_l^d)b_l^r+\sum_k(\partial w_0^d/\partial q_k^d)q_k^r \tag{2.24}$$

Since dispersion about mean is small and only first order terms are retained, derivatives of the random process have been approximated as the derivatives of their deterministic values.

The basic random variables b_l have been sequenced as follows:

$$b_1=E_{11},\ b_2=E_{22},\ b_3=\nu_{12} \text{ and } b_4=G_{12} \tag{2.25}$$

Using Taylor series expansion, keeping only one term in series and neglecting small quantities one can write the random part of reduced stiffness matrix and transformed reduced stiffness matrix as:

$$Q_{ij}^{r}=\sum_{l=1}^{p}(\partial(Q_{ij}^{d})/\partial b_{l}^{d})b_{l}^{r} \text{ and } \overline{Q}_{ij}^{r}=\sum_{l=1}^{p}(\partial(\overline{Q}_{ij}^{d})/\partial b_{l}^{d})b_{l}^{r} \tag{2.26}$$

The random part of extensional, bending-extension coupling and bending stiffness matrices can also be expressed similarly by using Taylor series expansion keeping only one term in the series and neglecting all higher order terms as:

$$A_{ij}^{r}=\sum_{l=1}^{p}(\partial(A_{ij}^{d})/\partial b_{l}^{d})b_{l}^{r}, \quad B_{ij}^{r}=\sum_{l=1}^{p}(\partial(B_{ij}^{d})/\partial b_{l}^{d})b_{l}^{r},$$

and $$D_{ij}^{r}=\sum_{l=1}^{p}(\partial(D_{ij}^{d})/\partial b_{l}^{d})b_{l}^{r} \tag{2.27}$$

where 'p' denotes the number of basic RV's.

The partial derivative of the reduced stiffness matrix with respect to each material property can be written as:

(i) Taking E_{11} as a random parameter

$$\partial Q_{11}^{d}/\partial E_{11}^{d}=E_{11}(E_{11}-2E_{22}\nu_{12}^{2})/(E_{11}-2E_{22}\nu_{12}^{2})^{2}$$

$$\partial Q_{22}^{d}/\partial E_{11}^{d}=-E_{22}^{2}\nu_{12}^{2}/(E_{11}-2E_{22}\nu_{12}^{2})^{2}$$

$$\partial Q_{12}^{d}/\partial E_{11}^{d}=-E_{22}^{2}\nu_{12}^{3}/(E_{11}-2E_{22}\nu_{12}^{2})^{2}$$

$$\partial Q_{66}^{d}/\partial E_{11}^{d}=0$$

(ii) Taking E_{22} as random parameter

$$\partial Q_{11}^{d}/\partial E_{22}^{d}=E_{11}^{2}\nu_{12}^{2}/(E_{11}-2E_{22}\nu_{12}^{2})^{2}$$

$$\partial Q_{22}^{d}/\partial E_{22}^{d}=E_{11}^{2}/(E_{11}-2E_{22}\nu_{12}^{2})^{2}$$

$$\partial Q_{12}^{d}/\partial E_{22}^{d}=E_{11}^{2}\nu_{12}/(E_{11}-2E_{22}\nu_{12}^{2})^{2}$$

$\partial Q_{66}^d / \partial E_{22}^d = 0$

(iii) Taking ν_{12} as random parameter (2.28)

$$\partial Q_{11}^d / \partial \nu_{12}^d = 2\, E_{11}^2\, E_{22}\, \nu_{12} / (E_{11} - 2E_{22}\, \nu_{12}^2)^2$$

$$\partial Q_{22}^d / \partial \nu_{12}^d = E_{11}\, E_{22}^2\, \nu_{12}^2 / (E_{11} - 2E_{22}\, \nu_{12}^2)^2$$

$$\partial Q_{12}^d / \partial \nu_{12}^d = E_{11}^2\, E_{22} + E_{11} E_{22}\, \nu_{12}^2 / (E_{11} - 2E_{22}\, \nu_{12}^2)^2$$

$$\partial Q_{66}^d / \partial \nu_{12}^d = 0$$

(iv) Taking G_{12} as random parameter

$$\partial Q_{11}^d / \partial G_{12}^d = 0$$

$$\partial Q_{11}^d / \partial G_{12}^d = 0$$

$$\partial Q_{12}^d / \partial G_{12}^d = 0$$

$$\partial Q_{66}^d / \partial G_{12}^d = 1$$

The partial derivatives of the transformed reduced stiffness matrix $\partial (\overline{Q}_{ij}^d)/\partial b_l^d$ can be found from equation (2.28) by using equation (2.2). Also the partial derivatives of the stiffness matrix elements A_{ij}^d, B_{ij}^d, D_{ij}^d with respect to 'b_l^d' are required for solving equation (2.27). These can be expressed as follows:

$$\partial (A_{ij}^d)/\partial b_l^d = \sum_{k=1}^{n} (\partial (\overline{Q}_{ij}^d)_k / \partial b_l^d)(h_k - h_{k-1})$$

$$\partial (B_{ij}^d)/\partial b_l^d = 1/2 \sum_{k=1}^{n} (\partial (\overline{Q}_{ij}^d)_k / \partial b_l^d)(h_k^2 - h_{k-1}^2) \tag{2.29}$$

$$\partial (D_{ij}^d)/\partial b_l^d = 1/3 \sum_{k=1}^{n} (\partial (\overline{Q}_{ij}^d)_k / \partial b_l^d)(h_k^3 - h_{k-1}^3)$$

Substituting equations (2.23) and (2.24) in equation (2.21) and collecting terms of different orders, we get zero order and first order equation as follows:

Zero order equation:

$$(L_1^d\ w_0^d + L_2^d\,(w_0^d)^2 + L_3^d\,(w_0^d)^3) = Q_0^d \tag{2.30}$$

First order equation:

$$(L_1^d + 2L_2^d\ w_0^d + 3L_3^d\,(w_0^d)^2)[\sum_l (\partial w_0^d / \partial b_l^d)\, b_l^r + \sum_k (\partial w_0^d / \partial q_k^d)\, q_k^r] + w_0^d \sum_l (\partial L_1^d / \partial b_l^d)\, b_l^r +$$

$$(w_0^d)^2 \sum_l (\partial L_2^d / \partial b_l^d)\, b_l^r + (w_0^d)^3 \sum_l (\partial L_3^d / \partial b_l^d)\, b_l^r = Q_0^r \tag{2.31}$$

Equation (2.30) gives the mean response of the system whereas equation (2.31) can be used to find the response variance of the system. In equation (2.31) $\partial w_0^d / \partial b_l^d$ and $\partial w_0^d / \partial q_k^d$ are the unknowns.

To find $\partial w_0^d / \partial b_l^d$, multiplying equation (2.31) by b_u^r and taking expectation we get the following under the assumption that the primary material variables and the loading components are independent random processes.

$$\{L_1^d + 2L_2^d\ w_0^d + 3L_3^d\,(w_0^d)^2\}(\partial w_0^d / \partial b_u^d) + (\partial L_1^d / \partial b_u^d)\, w_0^d + (\partial L_2^d / \partial b_u^d)(w_0^d)^2 +$$

$$(\partial L_3^d / \partial b_u^d)(w_0^d)^3 = 0 \quad (u=1, 2, \ldots, 4) \tag{2.32}$$

To find $\partial w_0^d / \partial q_k^d$, multiplying equation (2.31) by q_u^r and taking expectation we get the following with the assumption that b_l's and q_k's are independent of each other.

$$(L_1^d + 2L_2^d\ w_0^d + 3L_3^d\,(w_0^d)^2)(\partial w_0^d / \partial q_u^d) = 1 \quad (u=1) \tag{2.33}$$

Thereafter, the total response can be evaluated as follows:

$$w_0 = w_0^d + [\sum_l (\partial w_0^d / \partial b_l^d)\, b_l^r + \sum_k (\partial w_0^d / \partial q_k^d)\, q_k^r] \tag{2.34}$$

The variance of the static response takes the form:

$$\mathrm{Var}(w_0) = \mathrm{E}[\{\sum_l (\partial w_0^d / \partial b_l^d)\, b_l^r + \sum_k (\partial w_0^d / \partial q_k^d)\, q_k^r\}^2] \tag{2.35}$$

The SD is obtained as the square root of the variance. The above approach is general in nature, and can be applicable to a variety of problems in analysis of structures involving random system parameters and loading.

2.5 FREE VIBRATION ANALYSIS

2.5.1 System Equations

Consider a rectangular composite plate of constant total thickness 'h', composed of thin orthotropic layers bonded together and is executing free vibration.

In case of rectangular antisymmetric crossply all the coupling elements of [A], [B] and [D] matrices identically go to zero i.e. $A_{16}=A_{26}=B_{16}=B_{26}=D_{16}=D_{26}=0$. From equation (2.7) the strain energy for the plate can be written as:

$$U=\frac{1}{2}\int_0^a\int_0^b\{A_{11}\varepsilon_x^2+2A_{12}\varepsilon_x\varepsilon_y+A_{22}\varepsilon_y^2+A_{66}\gamma_{xy}^2+2B_{11}\kappa_x\varepsilon_x+B_{12}(\kappa_y\varepsilon_x+2\kappa_x\varepsilon_y)$$

$$+2B_{22}\kappa_y\varepsilon_y+2B_{66}\gamma_{xy}\kappa_{xy}+D_{11}\varepsilon_x^2+2D_{12}\kappa_x\kappa_y+D_{22}\varepsilon_y^2+D_{66}\kappa_{xy}^2\}dxdy \quad (2.36)$$

The kinetic energy of the plate, neglecting in-plane inertia is:

$$T=(1/2)\int_0^a\int_0^b(\sum\rho_i h_i)\,\dot{w}^2(x,y,t)\,dx\,dy \quad (2.37)$$

Assuming that the amplitude may be large, Von Karman non-linear strain-displacement is used to evaluate the frequency response of the system.

The following set of admissible functions satisfies the boundary conditions defined at equations (2.10):

$$u=u_0(t)Cos\frac{m\pi x}{a}Sin\frac{n\pi y}{b},\qquad v=v_0(t)Sin\frac{m\pi x}{a}Cos\frac{n\pi y}{b},$$

$$w=w_0(t)Sin\frac{m\pi x}{a}Sin\frac{n\pi y}{b} \quad (2.38)$$

where u, v and w are mid plane displacement in x, y and z directions and u_0 v_0 and w_0 are maximum displacements at any instant of time.

Substituting equations (2.8), (2.9) and (2.38) in equations (2.36) and (2.37) and integrating gives the total potential energy of the system:

$$\pi = U - T \tag{2.39}$$

Use of Hamilton's principle leads to two algebraic equations and one second-order ordinary differential equation in terms of u_0, v_0 and w_0.

$$T_1 u_0 + T_2 v_0 + T_3 w_0 + T_4 w_0^2 = 0 \tag{2.40}$$

$$T_2 u_0 + T_5 v_0 + T_6 w_0 + T_7 w_0^2 = 0 \tag{2.41}$$

$$(\Sigma \rho_i h_i)\, \ddot{w}_0 + T_3 u_0 + T_6 v_0 + T_8 w_0 + 2T_4 u_0 w_0 + 2T_7 v_0 w_0 + T_9 w_0^2 + T_{10} w_0^3 = 0 \tag{2.42}$$

where the coefficients T_1, T_2,...,T_{10} depend on the plate geometry, material properties and the mode shapes. Their expressions are defined in equation (2.19).

Substituting u_0 and v_0 in terms of w_0, obtained from the first two equations into the third equation results in a single second order differential equation containing quadratic and cubic nonlinear terms:

$$(\Sigma \rho_i h_i)\, \ddot{w}_0 + L_1 w_0 + L_2 w_0^2 + L_3 w_0^3 = 0 \tag{2.43}$$

where L_1, L_2 and L_3 are as follows:

$$\begin{aligned} L_1 &= [T_8 + \{(2T_2T_3T_6 - T_3^2T_5 - T_6^2T_1)/(T_1T_5 - T_2^2)\}] \\ L_2 &= [T_9 + \{(3T_2T_3T_7 + 3T_2T_4T_6 - 3T_3T_4T_5 - 3T_1T_6T_7)/(T_1T_5 - T_2^2)\}] \\ L_3 &= [T_{10} + \{(4T_2T_4T_7 - 2T_5T_4^2 - T_1T_7^2)/(T_1T_5 - T_2^2)\}] \end{aligned} \tag{2.44}$$

The energy balance equation can be obtained by multiplying equation (2.43) by $\dot{w}$ and integrating with respect to time:

$$(\Sigma \rho_i h_i)\, \dot{w}_0^2 + L_1 w_0^2 + (2/3) L_2 w_0^3 + (1/2) L_3 w_0^4 = H = const \tag{2.45}$$

When the plate is subjected to free vibration and undergo large amplitude oscillations, the constant H in equation (2.45) at $w_0=w_{max}$ i.e. $\dot{w}_0=0$, can be obtained as:

$$H = L_1 w_{\max}^2 + (2/3)L_2 w_{\max}^3 + (1/2)L_3 w_{\max}^4 \tag{2.46}$$

Substituting this constant H in equation (2.45) yields:

$$(\sum \rho_i h_i)\, \dot{w}_0^2 = L_1(w_{\max}^2 - w_0^2) + (2/3)L_2(w_{\max}^3 - w_0^3) + (1/2)L_3(w_{\max}^4 - w_0^4) \tag{2.47}$$

In the absence of coefficient L_2 equation (2.47) at $\dot{w}_0=0$ has two equal and opposite roots $\pm w_{max}$. Substituting for displacement response $w_0 = w_{max} Sin\,\theta$ the non-linear time period for such a plate can be obtained as:

$$T_{nl} = \frac{2\pi}{\omega} = 4\int_0^{\pi/2} \frac{d\theta\sqrt{\Sigma(\rho_i h_i)}}{\sqrt{[L_1 + \frac{1}{2}L_3(1+\sin^2\theta)w_{\max}^2]}} \tag{2.48}$$

On simplifying, one can rewrite the above equation as:

$$T_{nl} = \frac{4\sqrt{\Sigma(\rho_i h_i)}}{\sqrt{L_1(1+c)}}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\alpha\sin^2\theta}} \tag{2.49}$$

where $c=0.5L_3 w_{\max}^2/L_1$ and $\alpha=c/(1+c)$

Equation (2.49) is in a form of the elliptic integral, which cannot be evaluated in terms of elementary functions. An infinite series representation can be generated for the above integral by first expanding the integrands in binomial series and then using termwise integration:

$$\mathrm{T_{nl}} = \frac{2\pi\sqrt{\Sigma(\rho_i h_i)}}{\sqrt{L_1(1+c)}}[1 - \frac{\alpha}{2^2} + \frac{\alpha^2}{4^2} - \frac{\alpha^3}{8^2} + \frac{\alpha^4}{16^2} - ....] \tag{2.50}$$

The non-linear frequency can be expressed as:

$$\omega = 2\pi/T_{nl} \tag{2.51}$$

i.e. $\omega = \text{fun}(E_{11}, E_{22}, G_{12}, \nu_{12} \text{ and } w_{max})$ (2.52)

2.5.2 Mean and Variance of Dynamic Response – Perturbation Technique

Perturbation approach has been adopted for obtaining the second order frequency statistics with randomness in material properties of the flat plate. As stated earlier, it is assumed that the material properties are independent of each other and the dispersion of each random variable about its mean value is small.

Using Taylor series expansion the random part of the dependent variables is expressed in terms of the independent variables as in section 2.4.2. As b_l^r are assumed to be small in magnitude, the second and higher order terms are neglected and the expression may be put as:

$$L_i^r = \sum_l (\partial L_i^d / \partial b_l^d) b_l^r$$

$$\omega^r = \sum_l (\partial \omega^d / \partial b_l^d) b_l^r \tag{2.53}$$

Since dispersion about mean is small, total random process involved in the derivative have been approximated by its deterministic value. Hence from equation (2.52) we have

$\partial \omega^d / \partial b_l^d =$

$$\frac{\dfrac{\left(\dfrac{\partial L_1^d}{\partial b_l^d} + 0.5\dfrac{\partial L_3^d}{\partial b_l^d} w_{max}^2\right)}{2\sqrt{L_1^d + 0.5L_3^d w_{max}^2}}\left(1 - \dfrac{\alpha}{2^2} + \dfrac{\alpha^2}{4^2} - \ldots\right) - \sqrt{L_1^d + 0.5L_3^d w_{max}^2}\,\dfrac{d}{db_l^d}\left(1 - \dfrac{\alpha}{2^2} + \dfrac{\alpha^2}{4^2} - \ldots\right)}{\sqrt{\sum \rho_i h_i}\left(1 - \dfrac{\alpha}{2^2} + \dfrac{\alpha^2}{4^2} - \ldots\right)^2} \tag{2.54}$$

where partial derivative of L_1^d, L_3^d and α with respect to b_l^d can be expressed in terms of E_{11}, E_{22}, ν_{12} and G_{12} as presented for the static analysis.

The natural frequency can now be expressed as follows:

$$\omega = \omega^d + (\sum_l (\partial \omega^d / \partial b_l^d) b_l^r \tag{2.55}$$

The variance of the natural frequency can be put as:

$$\mathrm{Var}(\omega) = \mathrm{E}[\{\sum_l (\partial \omega^d / \partial b_l^d) b_l^r\}^2] \tag{2.56}$$

The SD is obtained as the square root of the variance.

2.6 FORCED VIBRATION ANALYSIS

2.6.1 System Equations

Consider a rectangular composite plate of constant total thickness 'h', composed of thin orthotropic layers bonded together. The plate is subjected to sinusoidal transverse dynamic loading given by

$$Q_0(x,y,t) = Q_0(t)\, Sin(\frac{m\pi x}{a})\, Sin(\frac{n\pi y}{b}) \tag{2.57}$$

where m and n are integers, a and b are the plate dimensions in x and y directions respectively and $Q_0(t)$ is the time dependent forcing function distribution amplitude. This loading form is generic and can be used to simulate different typesof loading on the plate.

In case of rectangular antisymmetric cross-ply all the coupling elements of [A], [B] and [D] matrices identically go to zero i.e. $A_{16}=A_{26}=B_{16}=B_{26}=D_{16}=D_{26}=0$. From equation (2.7) the strain energy for the plate can be written as:

$$U=\frac{1}{2}\int_0^a\int_0^b\{A_{11}\varepsilon_x^2+2A_{12}\varepsilon_x\varepsilon_y+A_{22}\varepsilon_y^2+A_{66}\gamma_{xy}^2+2B_{11}\kappa_x\varepsilon_x+B_{12}(\kappa_y\varepsilon_x+2\kappa_x\varepsilon_y)$$

$$+2B_{22}\kappa_y\varepsilon_y+2B_{66}\gamma_{xy}\kappa_{xy}+D_{11}\varepsilon_x^2+2D_{12}\kappa_x\kappa_y+D_{22}\varepsilon_y^2+D_{66}\kappa_{xy}^2\}dxdy \quad (2.58)$$

For a sinusoidal dynamic load defined in equation (2.57), the external work done can be represented by:

$$W=\int_0^a\int_0^b Q(x,y,t)\, w(x,y,t)\, dx\, dy \quad (2.59)$$

The kinetic energy of the plate, neglecting in-plane inertia is:

$$T=(1/2)\int_0^a\int_0^b (\Sigma\rho_i h_i)\, \dot{w}^2(x,y,t)dx\, dy \quad (2.60)$$

Assuming that the displacements may be large, Von Karman non-linear strain-displacement are used to evaluate the system response.

The following set of admissible functions satisfies the earlier defined boundary conditions:

$$u=u_0(t)\mathrm{Cos}\frac{m\pi x}{a}\mathrm{Sin}\frac{n\pi y}{b},\qquad v=v_0(t)\mathrm{Sin}\frac{m\pi x}{a}\mathrm{Cos}\frac{n\pi y}{b},$$

$$w=w_0(t)\mathrm{Sin}\frac{m\pi x}{a}\mathrm{Sin}\frac{n\pi y}{b} \quad (2.61)$$

where u, v and w are mid plane displacement in x, y and z directions and u_0 v_0 and w_0 are maximum displacements at any instant of time.

The strain energy, work done and kinetic energy can be obtained by substituting equations (2.8), (2.9) and (2.61) in equations (2.58)-(2.60) and performing the required integration.

Based on Hamilton's principle the variation of the Langrangian (*U-W -T*) provides the governing modal equation of motion.

$$(\Sigma\rho_i h_i)\ddot{w}_0 + L_1 w_0 + L_2 w_0^2 + L_3 w_0^3 = Q_0(t) \tag{2.62}$$

where L_1, L_2 and L_3 are as follows:

$$L_1=[T_8+\{(2T_2T_3T_6-T_3^2T_5-T_6^2T_1)/(T_1T_5-T_2^2)\}]$$

$$L_2=[T_9+\{(3T_2T_3T_7+3T_2T_4T_6-3T_3T_4T_5-3T_1T_6T_7)/(T_1T_5-T_2^2)\}] \tag{2.63}$$

$$L_3=[T_{10}+\{(4T_2T_4T_7-2T_5T_4^2-T_1T_7^2)/(T_1T_5-T_2^2)\}]$$

where $T_1, T_2, \ldots, T_{10}$ are coefficients depend on the plate geometry, material properties and the mode shape. These expressions are defined in equation (2.19).

The energy balance equation is obtained by multiplying equation (2.62) by $\dot{w}$ and integrating with respect to time:

$$(\Sigma\rho_i h_i)\dot{w}_0^2 + L_1 w_0^2 + (2/3)L_2 w_0^3 + (1/2)L_3 w_0^4 - Q_0(t)\, w_0 = H = \text{const} \tag{2.64}$$

When the plate is subjected to a transverse loading $Q(x,y,t)$ and undergo large amplitudes, the constant in equation (2.64) at $w_0 = w_{max}$, that is, $\dot{w}_0 = 0$, can be obtained as

$$H = L_1 w_{max}^2 + (2/3)L_2 w_{max}^3 + (1/2)L_3 w_{max}^4 - Q_0 w_{max} \tag{2.65}$$

Substituting this constant H in equation (2.64), yields:

$$(\Sigma\rho_i h_i)\dot{w}_0^2 + L_1(w_{max}^2 - w_0^2) + (2/3)L_2(w_{max}^3 - w_0^3) + (1/2)L_3(w_{max}^4 - w_0^4)$$

$$-(Q_0 w_{max} - Q_0(t)\, w_0) = 0 \tag{2.66}$$

n the absence of coefficient L_2 equation (2.66) at $\dot{w}_0 = 0$ has two equal and opposite roots w_{max}. Substituting $w_0 = w_{max} \, Sin\theta$ and $Q_0(t) = Q_0 \, Sin\theta$ the nonlinear time period for ıch a plate can be obtained as:

$$_{nl} = \frac{2\pi}{\omega} = 4\int_0^{\pi/2} \frac{d\theta\sqrt{\Sigma(\rho_i h_i)}}{\sqrt{[L_1 + \frac{1}{2}L_3(1+\sin^2\theta)w_{max}^2 - \frac{Q_0}{w_{max}}]}} \tag{2.67}$$

he above equation can be written as:

$$T_{nl}=4\int_0^{\pi/2} \frac{d\theta\sqrt{\Sigma(\rho_i h_i)}}{\sqrt{L_1}\sqrt{[1+c-d+b\sin^2\theta]}} \tag{2.68}$$

On further simplification, it can be put as:

$$T_{nl}=\frac{4\sqrt{\Sigma(\rho_i h_i)}}{\sqrt{L_1(1+c-d)}}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\alpha\sin^2\theta}} \tag{2.69}$$

where $c=0.5L_3 w_{\max}^2/L_1$, $d=Q_0/(L_1 w_{max})$ and $\alpha=c/(1+c-d)$

The above integral is in the term of an elliptic integral, which can not be evaluated in terms of elementary functions. An infinite series representation for above integral is generated, as in the previous section, by first expanding the integrands in binomial series and then using termwise integration. The final expression can be written as

$$T_{nl}=\frac{2\pi\sqrt{\Sigma(\rho_i h_i)}}{\sqrt{L_1(1+c-d)}}[1-\frac{\alpha}{2^2}+\frac{\alpha^2}{4^2}-\frac{\alpha^3}{8^2}+\frac{\alpha^4}{16^2}-...] \tag{2.70}$$

The above equation can also be rearranged to give the unknown w_{max} in terms of the known forcing frequency ω. This is a fifth order nonlinear algebraic equation given by:

$$A\,w_{\max}^5+B\,w_{\max}^3+C\,w_{\max}^2+D\,w_{max}+E=0 \tag{2.71}$$

where $A=50\,L_3^2/64$, $B=L_1L_3-L_3\omega^2\Sigma\rho_i h_i$, $C=(50/32)(L_1L_3-L_3P_0)$,

$D=2\,L_1^2-2L_1\omega^2\Sigma\rho_i h_i$ and $E=2P_0L_1$

Hence the non-linear amplitude can be written as

i.e. $w_{max}=\text{fun}(E_{11}, E_{22}, G_{12}, \nu_{12}, \omega$ and $Q_0)$ (2.72)

Equation (2.72) gives the required non-linear amplitude for the specified problem.

2.6.2 Mean and variance of forced vibration response – perturbation technique

Perturbation approach has been adopted for obtaining the deflection or amplitude statistics with randomness in material properties and excitation for a flat plat. Here also, it is assumed that all the material properties and loading components are independent of each other. It is further assumed that the dispersion of each random variable about its mean value is small. The random variable may be split up as the sum of mean variable and zero mean random part with generality as in equation (2.23)

Using Taylor series expansion some of the terms of the equation (2.71) can be expanded as:

$$L_i^r = \sum_l (\, \partial L_i^d / \partial b_l^d \,) b_l^r$$

$$w_{\max}^r = \sum_l (\, \partial w_{\max}^d / \partial b_l^d \,) b_l^r + \sum_k (\, \partial w_{\max}^d / \partial q_k^d \,) q_k^r \qquad (2.73)$$

Assuming b_l^r to be small in magnitude, the second higher order terms are neglected. Here q_k^r are RV components of applied loading. Since dispersion about mean is small, total andom process involved in the derivative process can be taken as the derivative of the eterministic values.

Using equation (2.72) we get the derivative of $w_{\max}^d$ with respect of b_l^d and q_k^d as ound in static case. We have:

$$\partial w_{\max}^d / \partial b_l^d = \frac{\dfrac{\partial E}{\partial b_l^d} - \dfrac{\partial A}{\partial b_l^d} w_{\max}^5 - \dfrac{dB}{db_l^d} w_{\max}^3 - \dfrac{\partial C}{\partial b_l^d} w_{\max}^2 - \dfrac{\partial D}{\partial b_l^d} w_{\max}}{5Aw_{\max}^5 + 3Bw_{\max}^2 + 2Cw_{\max} + D} \qquad (2.74)$$

imilarly $\partial w_{\max}^d / \partial q_k^d$ can be found as:

$$\partial w_{\max}^{d} / \partial q_{k}^{d} = \frac{\dfrac{\partial E}{\partial q_{k}^{d}} - \dfrac{\partial C}{\partial q_{k}^{d}} w_{\max}^{2}}{5Aw_{\max}^{5} + 3Bw_{\max}^{2} + 2Cw_{\max} + D} \tag{2.75}$$

where partial derivative of A, B, C, D and E with respect to b_l^d and q_k^d can be expressed in terms of E_{11}, E_{22}, ν_{12} and G_{12}.

Now the total deflection response can be evaluated as follows:

$$w_{\max} = w_{\max}^{d} + (\sum_{l} (\partial w_{\max}^{d} / \partial b_{l}^{d}) b_{l}^{r} + (\partial w_{\max}^{d} / \partial q_{k}^{d}) q_{k}^{r}) \tag{2.76}$$

The variance of the response can be expressed by:

$$\mathrm{Var}(w_{\max}) = \mathrm{E}[(\sum_{l} (\partial w_{\max}^{d} / \partial b_{l}^{d}) b_{l}^{r} + (\partial w_{\max}^{d} / \partial q_{k}^{d}) q_{k}^{r})^{2}] \tag{2.77}$$

The SD is the square root of the variance. The above approach is general in nature, and can be applicable to different forced vibration problems involving random system parameters and loading.

CHAPTER 3

RESULTS AND DISCUSSION

3.1 INTRODUCTION

In this chapter numerical results have been presented and the behaviour of composite flat plates with random material properties and random loading has been described. This chapter is divided into three sections. First section describes the nonlinear behaviour of static deflection, the second section describes the nonlinear free vibration whereas the third section describes the nonlinear forced vibration behaviour. All edges simply supported boundary condition is used in all the three section.

Closed form expressions have been developed for the mean and variance of response for simply supported plates in the previous chapter. These expressions are used to generate the numerical results. Results are obtained for symmetric and antisymmetric cross ply laminates. All lamina are assumed to have the same thickness and the material properties are orthotropic along the material axes. The validation of the developed expressions is checked with available results in literature. The effects of material property dispersion along with variations in aspect ratio, thickness ratio and oscillation amplitude on the frequency statistics have been studied.

3.2 STATIC DEFLECTION

3.2.1 Validation

The validation of the present formulation is sought by comparison of results with reported literature. The comparison has to be made with the linear formulation only, as

nonlinear formulation and results are not available for laminated composite plate with random material properties and loading. Based on equations (2.32), (2.33) and (2.35) deflection variance for linear strain-displacement relation obtained by taking $L_2=L_3=0$, with all basic material properties varying simultaneously are compared with the result by Salim et al. [10]. Table 3.1 represents the comparison of SD of center point deflection for four layered [0°/90°/90°/0°] square symmetric cross ply with thickness ratio $b/h=100$. The mean value of the material properties and loading used are [10]:

$E_{11}=181$ GPa, $E_{22}=10.3$ GPa, $G_{12}=7.17$ GPa, $\nu_{12}=0.28$ and $Q_0=1.65 \times 10^3$ Pa

where E_{11} and E_{22} denote longitudinal and transverse elastic modulus, respectively, G_{12} is the inplane shear modulus and ν_{12} denotes the Poisson's ratio. The plate edges are simply supported and the mid point deflection is non-dimensionlised with the plate thickness. A reasonable good agreement between the two is observed. The results from the present study with nonlinear formulation are also placed in the table for comparison. The present approach is expected to give more accurate results as it accounts for non-linearity in the system equation. Figure 3.1 presents the comparison of the linear response with nonlinear cases for different loadings. It can be seen that when loading is increased, the linear approach loses accuracy as compared to the present approach.

3.2.2 Second Order Deflection Statistics

The second order statistics of transverse deflection has been obtained for symmetric and anti-symmetric cross-ply rectangular composite plate with all edges simply supported. The laminates are subjected to a transverse sinusoidal load. For graphite/epoxy composite plate the mean values of the variables considered as random

for the analysis are – the material properties E_{11}=25E_{22}, G_{12}=0.5 E_{22}, ν_{12}=0.25 and the external loading Q_0=2.0 × 10^3 Pa

The influence of scattering in the material properties on the static deflection has been obtained by allowing the coefficient of variation to change from 0 to 20 % for laminated cross ply plates. The plates have aspect ratios b/a=1, 2, 3 and 4 and thickness ratio 100 with stacking sequences of [0°/90°/90°/0°] and [0°/90°/0°/90°].

3.2.2.1 Mean Deflection – Table 3.2 presents the non-dimensionalised mean central deflection for the four-layered symmetric laminate with different aspect ratios. Some available results by Singh et al. [8] are also placed in the table for comparison. The reference uses Rayleigh-Ritz method for the solution. The present approach gives an exact solution for the mean analysis. A reasonable good agreement between the two is observed.

Table 3.3 presents the non-dimensionalised mean value of the central deflection for different plate thickness with simply supported edges. The mean deflection, as expected, decreases with increase in the plate thickness.

3.2.2.2 Variance of deflection – The variation of non-dimensionalised deflection with SD of all the basic inputs changing simultaneously for three cases--square symmetric, square anti-symmetric and rectangular symmetric four layered cross-ply with a/b=2 is represented in Figure 3.2. It is seen that the change in deflection for square symmetric cross-ply is greater than the other two cases. Figures 3.3 (a)-(d) present deflection sensitivity to dispersion in only one basic material variable at a time for b/h=100. The

variations of deflection are most affected by changes in E_{11}. It is observed that the deflection is least affected with dispersion in ν_{12}. Figure 3.4 shows deflection sensitivity to dispersion in load only. It is seen that variation of load has a dominant effect on the deflection scattering as compared to E_{22}, G_{12} and ν_{12}.

Change in SD of deflection due to dispersion in all the basic random inputs changing simultaneously for symmetric cross-ply with different aspect ratio a/b=1, 2, 3 and 4 are presented in Figure 3.5. The effect on the scattering of deflection due to all basic random inputs decreases with increase in the aspect ratio.

Figure 3.6 shows midpoint deflection sensitivity to dispersion in all the basic random inputs changing simultaneously for different plate thickness ratios (b/h)=100, 50 and 33.33. The effect on the scattering of deflection increases with increase in thickness whereas for the linear case it is independent of variations in thickness.

Figure 3.7(a)-(b) show variation in deflection SD with dispersion in all the basic random inputs changing simultaneously for square anti-symmetric and rectangular symmetric plates with different thickness ratios of 100, 50 and 33.33. Figure 3.8 (a)–(c) show deflection sensitivity to dispersion in all the basic random inputs changing simultaneously with different loading for square symmetric, square antisymmetric and rectangular symmetric cross ply. A comparison with linear analysis with same boundary condition and loading is also presented. It is observed that the effect on scattering of deflection decreases with increase in loading whereas for linear case it is insensitive to change in mean loading.

Table 3.1: Comparison of SD/h of center point deflection for $[0^\circ/90^\circ/90^\circ/0^\circ]$ laminate with a/b=1 and thickness ratio b/h=100

SD/mean of material properties	0.05	0.10	0.15
(SD/h, w_0)			
Salim [6](Linear)	0.00360	0.00715	0.01080
Present Study (Linear)	0.00371	0.00724	0.01091
Present Study (Non-Linear)	0.00328	0.00690	0.01003

Table 3.2: Nondimensionalised mean central deflection (w_{nl}/h) for $[0^\circ/90^\circ/90^\circ/0^\circ]$ laminate with different aspect ratios and b/h=100

	Aspect Ratio (a/b)				
	a/b=1	a/b=2	a/b=3	a/b=4	a/b=5
Present work	0.518	0.8124	0.8435	0.8503	0.8527
Ref [13]	0.521	0.8128			

Table 3.3: Nondimensionalised mean value of the central deflection (w_{nl}/h) for different plate thickness

	Thickness ratio (b/h)			
	100	50	33.33	25
(a) Stacking sequence:[$0^\circ/90^\circ/90^\circ/0^\circ$]:				
a/b=1	0.5180	0.1069	0.0319	0.0135
a/b=2	0.8124	0.4077	0.1493	0.0638
(b) Stacking sequence:[$0^\circ/90^\circ/0^\circ/90^\circ$]:				
a/b=1	0.5431	0.1250	0.0374	0.0158

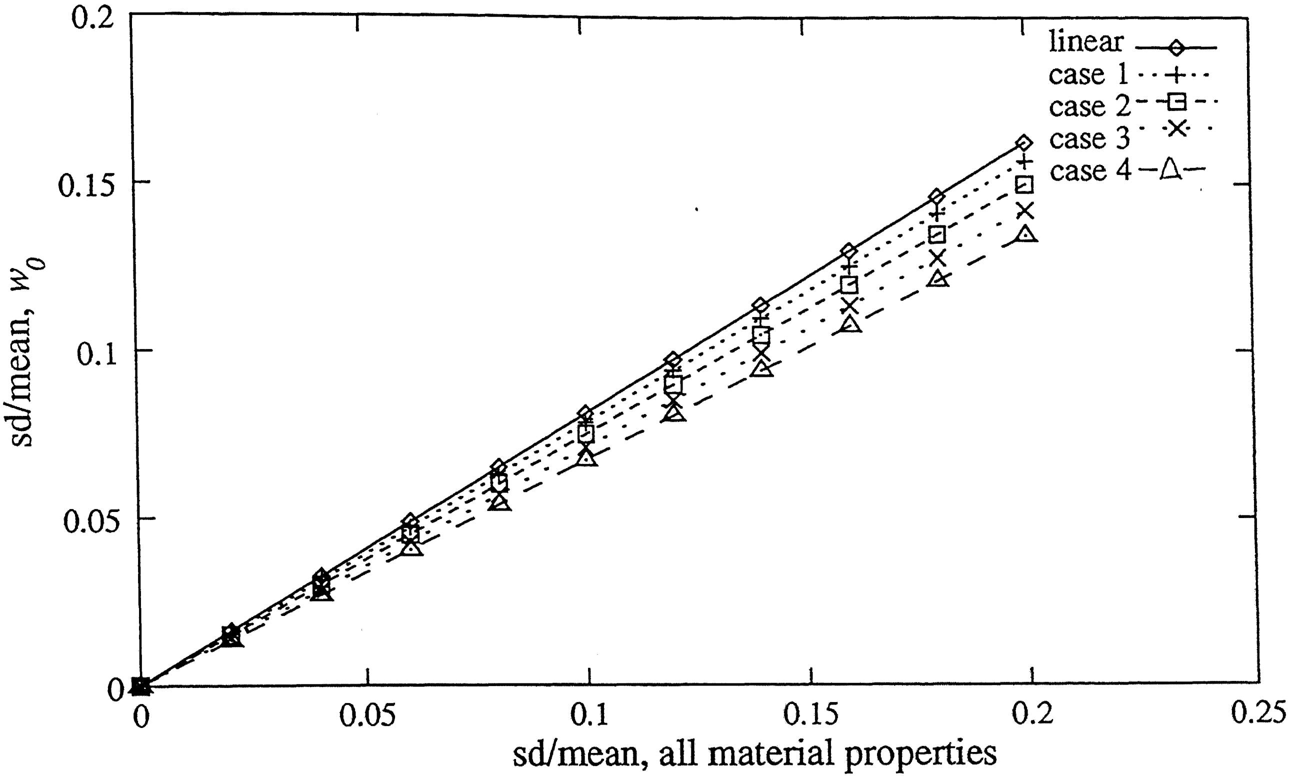


Figure 3.1- Comparison for coefficients of variation of deflection of [$0^0/90^0/90^0/0^0$] laminate having a/b=1 and b/h=100 with all material properties changing simultaneously. Case1 - Q_0=1.65 × 10^3 Pa, Case2 - Q_0=2.65 × 10^3 Pa, Case3 - Q_0=3.65 × 10^3 Pa, Case4 - Q_0=4.65 × 10^3 Pa

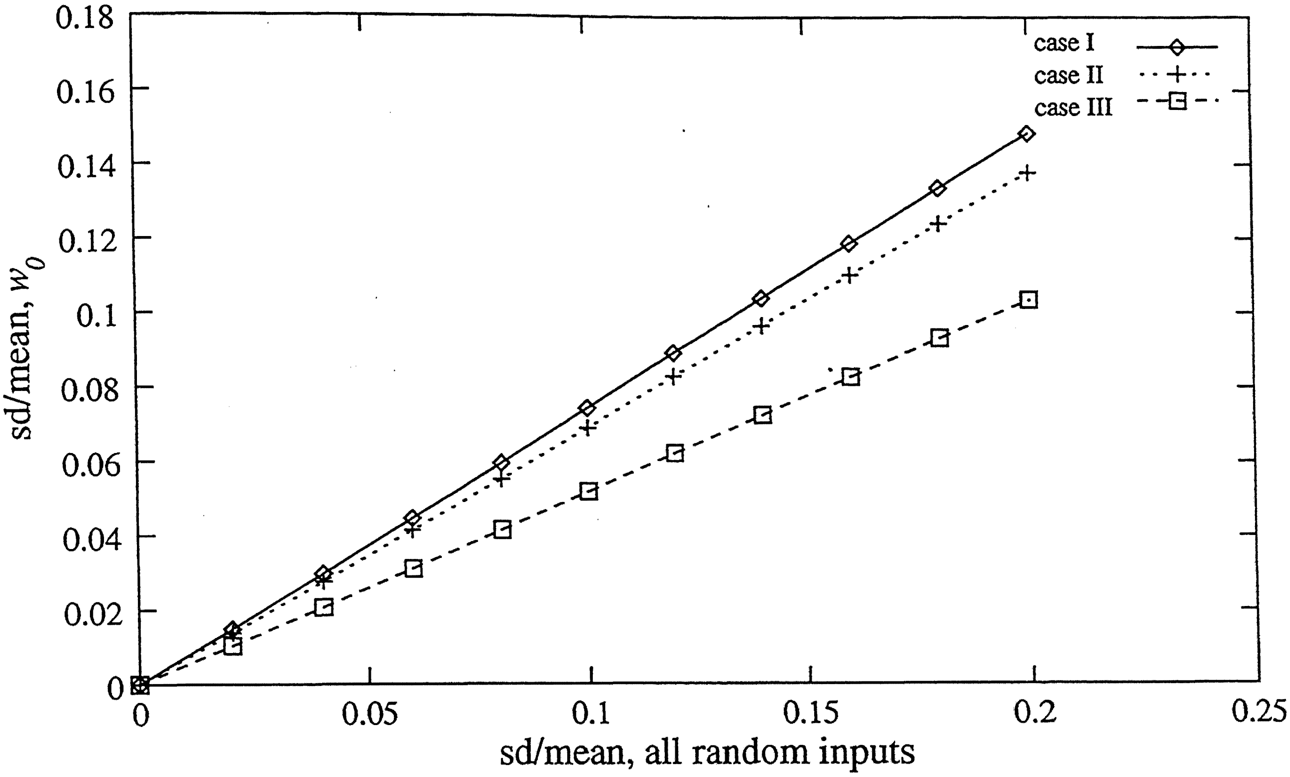


Figure 3.2- Influence of SD of all basic random inputs changing simultaneously on coefficient of variation of deflection for different cross-ply with b/h=100.
CaseI-[$0^0/90^0/90^0/0^0$] laminate with a/b=1, CaseII-[$0^0/90^0/0^0/90^0$] laminate with a/b=1, CaseIII- [$0^0/90^0/90^0/0^0$] laminate with a/b=2.0

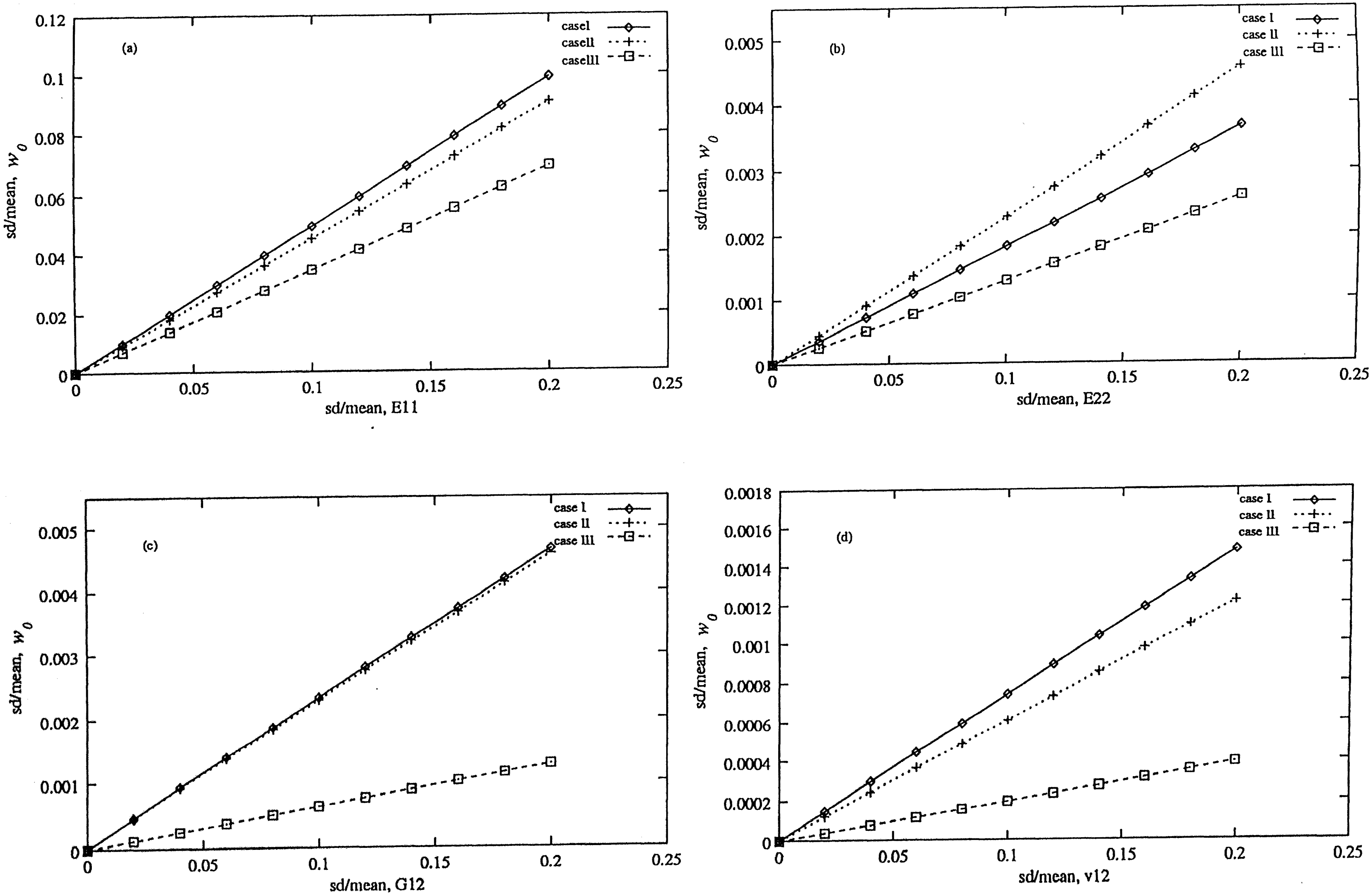


Figure 3.3- Influence of SD of basic material properties on coefficient of variation of deflection of different cross ply with b/h=100, (a) only E_{11} varying, (b) only E_{22} varying, (c) only G_{12} varying, (d) only ν_{12} varying. (Three cases same as in Figure 3.2)

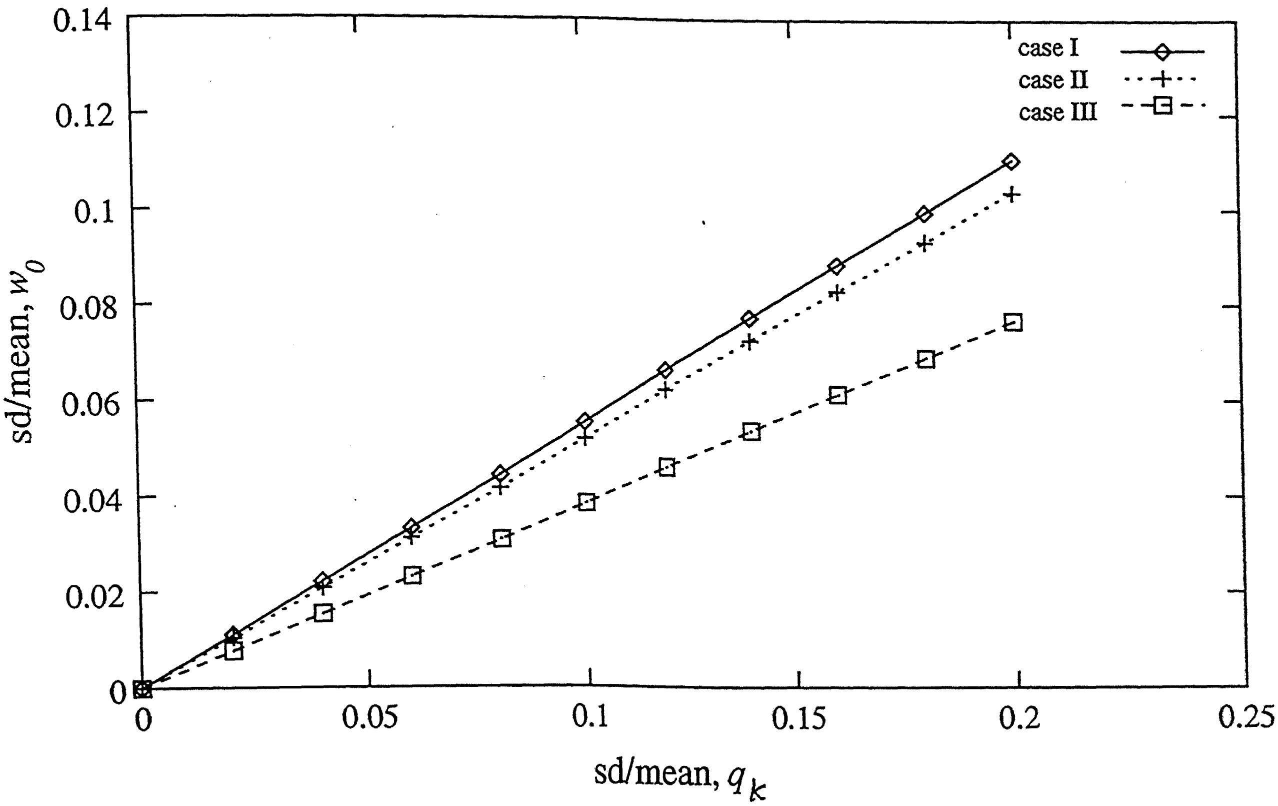


Figure 3.4- Influence of SD of load 'q_k' on coefficient of variation of deflection for different cross ply with b/h=100. (Three cases same as in Figure 3.2)

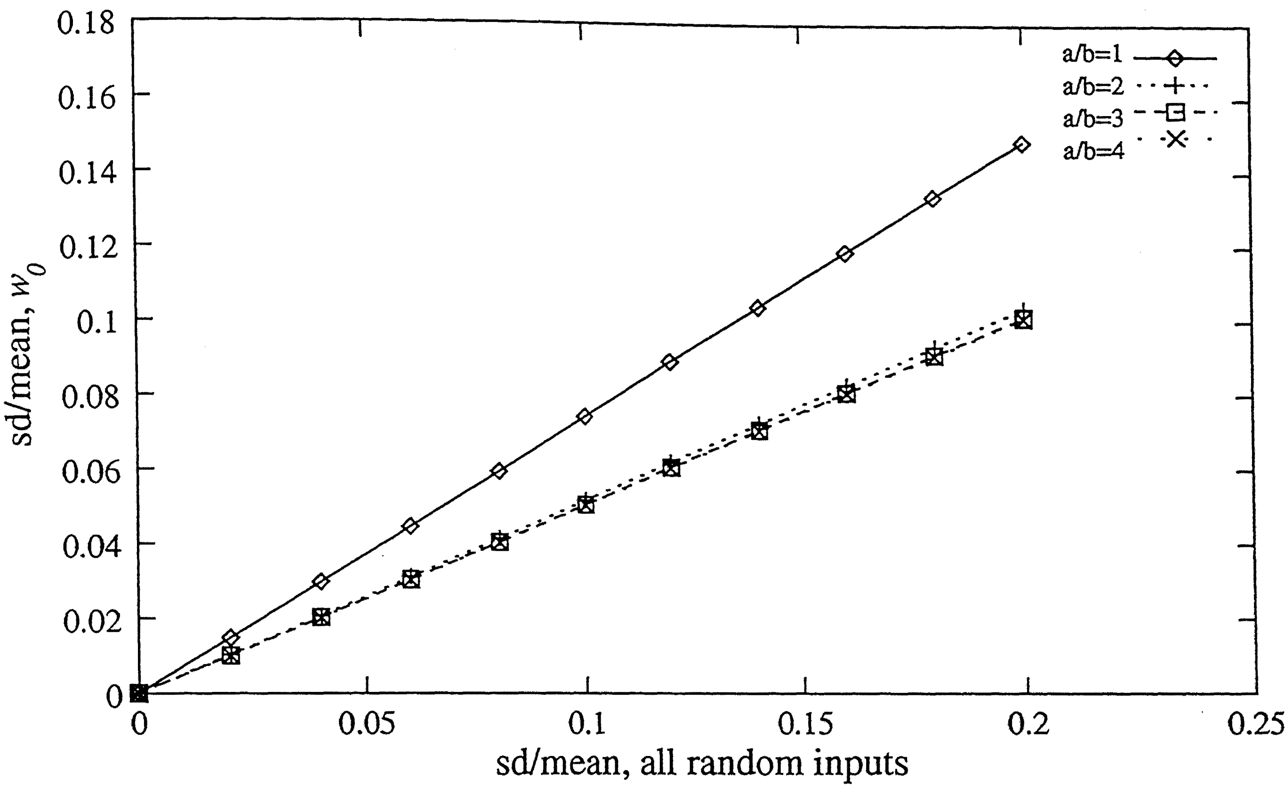


Figure 3.5- Influence of SD of all basic random inputs changing simultaneously on coefficient of variation of deflection for $[0^0/90^0/90^0/0^0]$ laminate having different aspect ratios

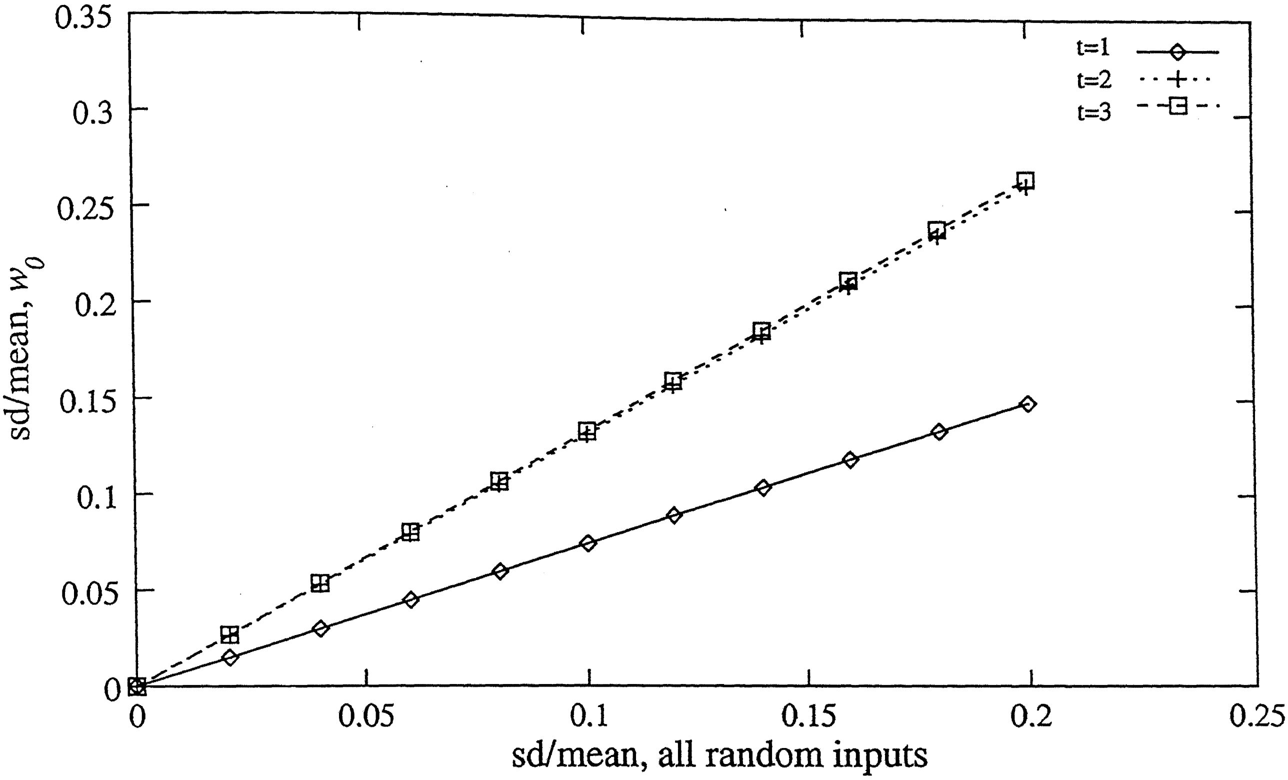


Figure 3.6- Influence of SD of all basic random inputs changing simultaneously on coefficient of variation of deflection for $[0^0/90^0/90^0/0^0]$ laminate having different plate thickness with a/b=1

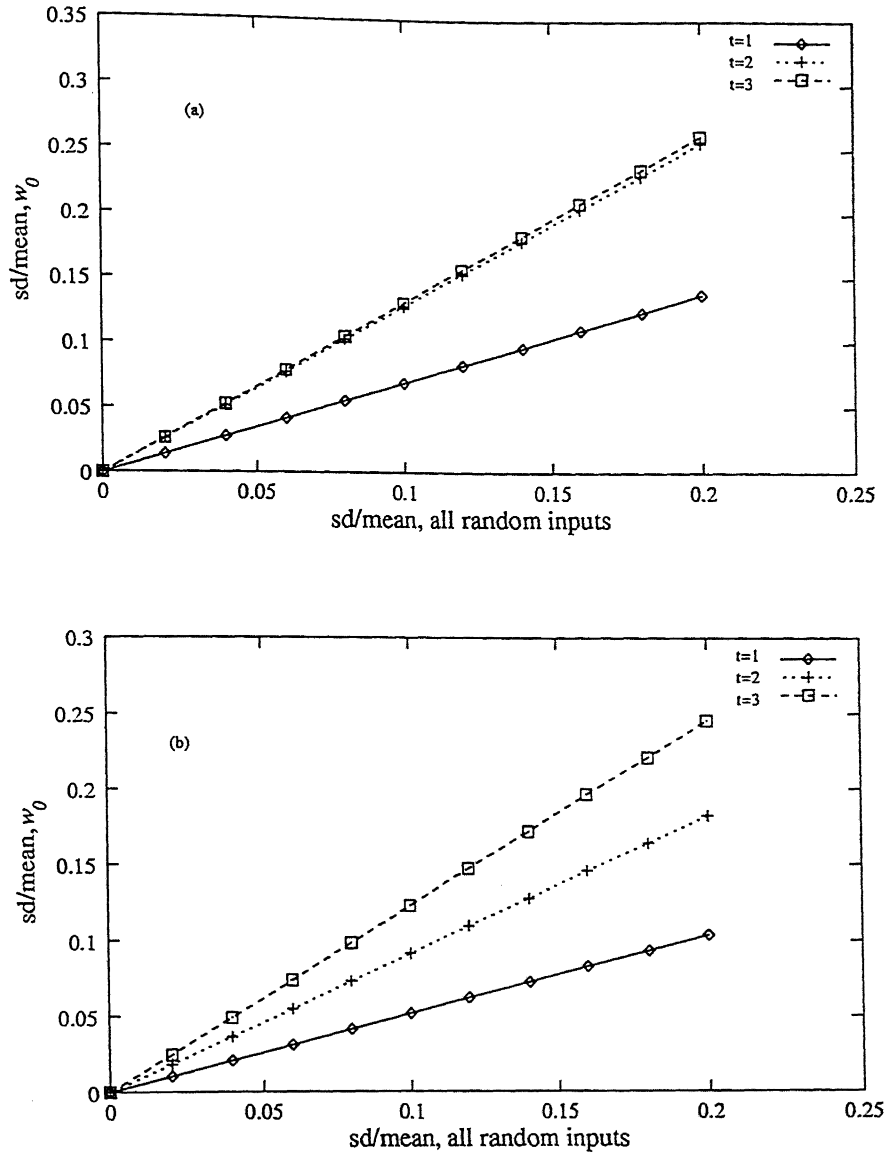
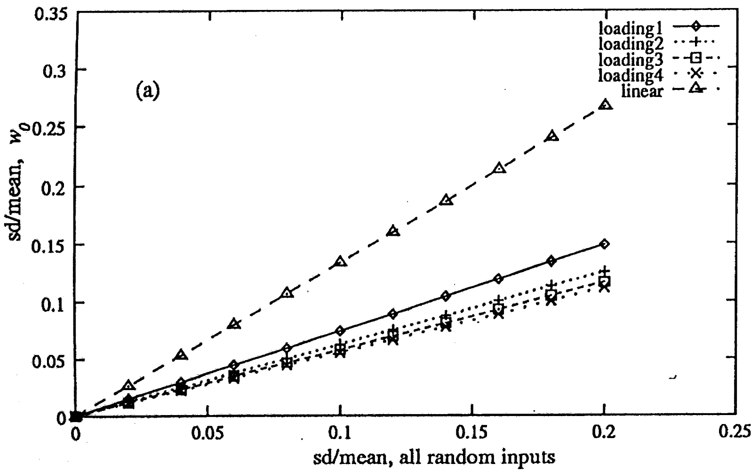
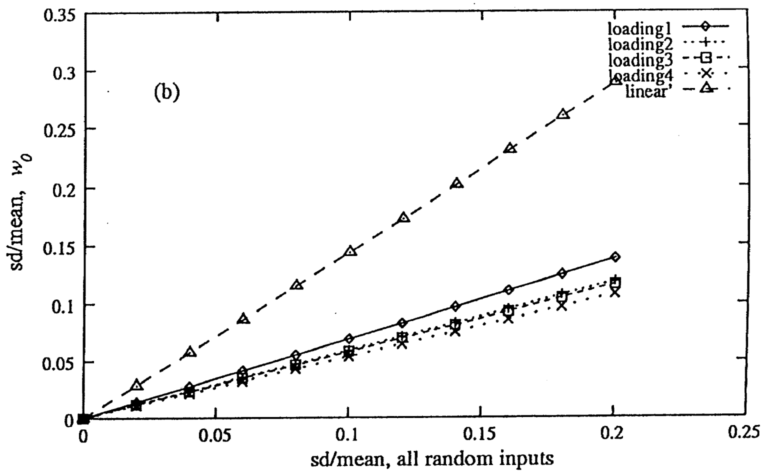


Figure 3.7- Influence of SD of all basic random inputs changing simultaneously on coefficient of variation of deflection of different cross-ply with different plate thickness for (a) [$0^0/90^0/0^0/90^0$] laminate with a/b=1, (b) [$0^0/90^0/90^0/0^0$] laminate with a/b=2





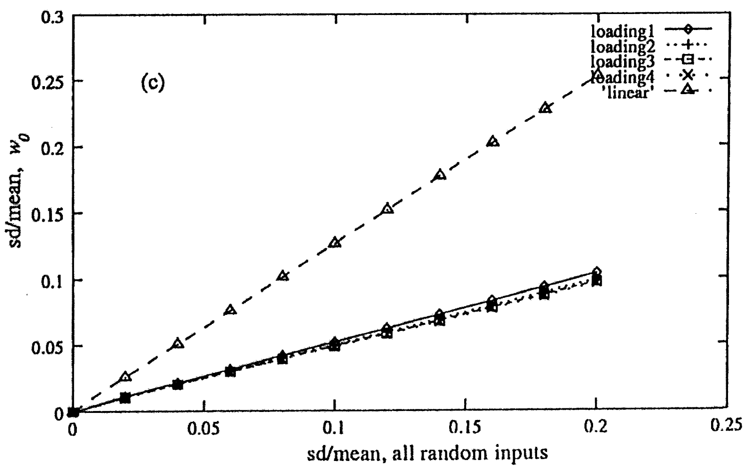


Figure 3.8- Influence of SD of all basic random inputs changing simultaneously on coefficient of variation of deflection of different cross-ply having different loading on the plate with b/h=100. (a) $[0^0/90^0/90^0/0^0]$ laminate with a/b=1, (b) $[0^0/90^0/0^0/90^0]$ laminate with a/b=1, (c) $[0^0/90^0/90^0/0^0]$ with a/b=2; loading 1- Q_0=2.0 × 10^3 Pa, loading 2- Q_0=4.0 × 10^3 Pa, loading 3- 6.0 × 10^3 Pa, loading4 - 8.0 × 10^3 Pa.

3.3 FREE VIBRATION

3.3.1 Validation

The validation of the present formulation is sought by comparison of results with reported literature. However, nonlinear formulation is not available in literature for laminated composite plate with random material properties hence comparison has been made with the linear formulation only with the linear form as a special case of the present nonlinear formulation. Based on equations (2.54) and (2.56) frequency variance for linear strain-displacement relation obtained by taking $L_3=0$, with all basic material properties varying simultaneously are compared with the result by Singh et al. [25]. Table 3.4 represents the comparison of SD of frequency for four layered [0°/90°/90°/0°] square symmetric cross ply with thickness ratio $b/h=10$. The mean values of the material properties used are [25]:

$$E_{22}=10.3 \text{ GPa}, \quad E_{11}=25E_{22}, \quad G_{12}=0.5E_{22}, \quad \text{and} \quad \nu_{12}=0.25$$

The plate edges are simply supported and the SD of frequency is non-dimensionalised with the mean frequency. A reasonable good agreement between the two is observed. The results from the present study with nonlinear formulation are also placed in the table for comparison. The effect of nonlinearity is apparent with different value of the amplitude w_{max}.

Table 3.5 presents a comparison of the non-dimensionalised mean frequency for the four-layered symmetric laminate with aspect ratio a/b= 2 having different amplitudes with limited available results by Singh et al. [14]. The material properties used for the analysis are [14]:

$$E_{11}=40E_{22}, \quad G_{12}=0.5E_{22}, \quad \text{and} \quad \nu_{12}=0.25.$$

The reference uses direct numerical integration method for the analysis of mean frequency whereas the present approach gives an exact solution. A reasonably good agreement between the two is observed.

3.3.2 Second order frequency statistics

The material used for the graphite/epoxy composite plate is the same as employed for generating Table 3.5. All the four material properties are considered as random for the analysis.

3.3.2.1 Mean frequency

Table 3.6 presents the non-dimensionalised mean nonlinear frequency for different plate thickness and amplitudes with simply supported edges. Influence of the scattering in the material properties on the mean frequency has been obtained by allowing the coefficient of variation to change from 0 to 20 % for laminated cross ply plates. The plates have aspect ratios b/a=1 and 2 and thickness ratios b/h = 100, 50 and 33.33 with stacking sequences of [0°/90°/90°/0°] and [0°/90°/0° 90°]. The mean nondimensional frequency decreases with increase in the plate thickness and increases with the oscillation amplitude. The anti-symmetric laminate has higher mean frequency compared to the symmetric laminate.

3.3.2.2 Variance of frequency

The variations of non-dimensionalised frequency with dispersion in all the basic material properties changing simultaneously are presented in Figures 3.9(a) - (c). Three cases square symmetric, square anti-symmetric and rectangular (a/b=2) symmetric four

layered cross ply with b/h=100 have been examined. It is seen that as normalised amplitude (w'=100 w_{max}/b) increases the frequency sensitivity increases for all cases. The sensitivity of the square anti-symmetric cross ply is greater than the symmetric case. The frequency dispersion also increases with increase in the aspect ratio.

Figures 3.10 (a) - (d) present the frequency sensitivity to dispersion with only one basic variable random at a time with different amplitudes for square symmetric cross ply for b/h=100. The frequency variations are most affected by change in E_{11} and least affected by dispersion in ν_{12}. It is also seen that sensitivity of the plate frequency to oscillation amplitude increases with increased variations in E_{11} and decreases with E_{22}, ν_{12} and G_{12}. Figures 3.11 (a) - (d) and Figures 3.12 (a) - (d) show the frequency sensitivity to dispersion for only one basic variable random at a time with different amplitudes for square antisymmetric and rectangular symmetric cross ply respectively for b/h=100.

Frequency sensitivity to dispersion in all the basic random inputs changing simultaneously for symmetric cross-ply with aspect ratio a/b=1, 2, 3 and 4 are presented in Figure 3.13(a)-(d). The effect on the scattering of frequency increases with increase in the aspect ratio from 1 to 2, decreases slightly from 2 to 3 and then remains almost constant for aspect ratios greater than 3.

Figure 3.14(a)-(c) shows changes in frequency SD due to dispersion in all the basic random inputs changing simultaneously for square symmetric cross-ply with different amplitude and plate thickness ratios (b/h)=100, 50 and 33.33. The effect on frequency scatter shows an initial decrement with increase in thickness ratio. This effect is less pronounced for thicker plates. For the linear case, frequency is independent of variations in both thickness ratios and oscillation amplitudes.

Figures 3.15(a)-(c) and 3.16(a)-(c) show frequency sensitivity to dispersion in all the basic random inputs changing simultaneously for square antisymmetric and rectangular symmetric cross-ply with different amplitude and plate thickness ratios (b/h)=100, 50 and 33.33. The effects on frequency scatter for both square anti-symmetric and rectangular symmetric laminate due to thickness ratio has the same nature as that of the symmetric cross-ply.

Influence of amplitude (w_{max}/h) on frequency coefficient of variation with a constant SD/mean for all material properties changing simultaneously for symmetric cross-ply with b/h=100 is shown in Figure 3.17. It is found that the frequency scatter increases nonlinearly with increase in amplitude and also increases with increase in variation of material properties.

3.4 FORCED VIBRATION

3.4.1 Validation

The validation of the present formulation is sought by comparison of results with reported literature. However, nonlinear formulation is not available in literature for laminated composite plate with random material properties and random loading hence comparison has been made with the mean analysis only.

Table 3.7 presents a comparison of the non-dimensionalised mean frequency for the square two-layered cross-ply laminate with different amplitude with limited available results by Singh et al. [19]. The material properties used for the analysis are [19]:

$E_{11}=40E_{22}$, $G_{12}=0.5E_{22}$, and $\nu_{12}=0.25$, and excitation $Q_0=0.2\,L_1$

Table 3.4: Comparison of SD(ω^2)/mean(ω^2) for [0°/90°/90°/0°] laminate with *a/b*=1 and thickness ratio *b/h*=10

SD/mean of material properties	0.05	0.10	0.15	0.20
(SD(ω^2)/mean(ω^2)				
Singh [6](Linear)	0.045	0.091	0.138	0.180
Present Study (Linear)	0.044	0.088	0.133	0.177
Present Study (w'=0.3) (Non-Linear)	0.048	0.097	0.146	0.194
(w'=0.6)	0.051	0.103	0.154	0.207
(w'=0.9)	0.054	0.109	0.162	0.221

Table 3.5: Comparison of the nondimensionalised mean nonlinear frequency (ω_{nl}/ω_l) for [0°/90°/90°/0°] laminate with aspect ratio a/b= 2 and *b/h*=100

	w_{max}=0.3	w_{max}=0.6	w_{max}=0.9
Present work	1.18	1.61	2.16
Ref [13]	1.22	1.63	2.18

Table 3.6: Nondimensionalised mean nonlinear frequency (ω_{nl}/ω_l) for different plate thickness and amplitudes

		Thickness ratio (b/h)		
		100	50	33.33
(a) Stacking sequence: $[0^\circ/90^\circ/90^\circ/0^\circ]$:				
a/b=1	(w'=0.3)	1.082	1.021	1.009
	(w'=0.6)	1.300	1.082	1.037
	(w'=0.9)	1.602	1.178	1.082
a/b=2	(w'=0.3)	1.204	1.054	1.024
	(w'=0.6)	1.680	1.204	1.095
	(w'=0.9)	2.265	1.421	1.204
(b) Stacking sequence: $[0^\circ/90^\circ/0^\circ/90^\circ]$:				
a/b=1	(w'=0.3)	1.097	1.025	1.011
	(w'=0.6)	1.351	1.097	1.044
	(w'=0.9)	1.695	1.209	1.097

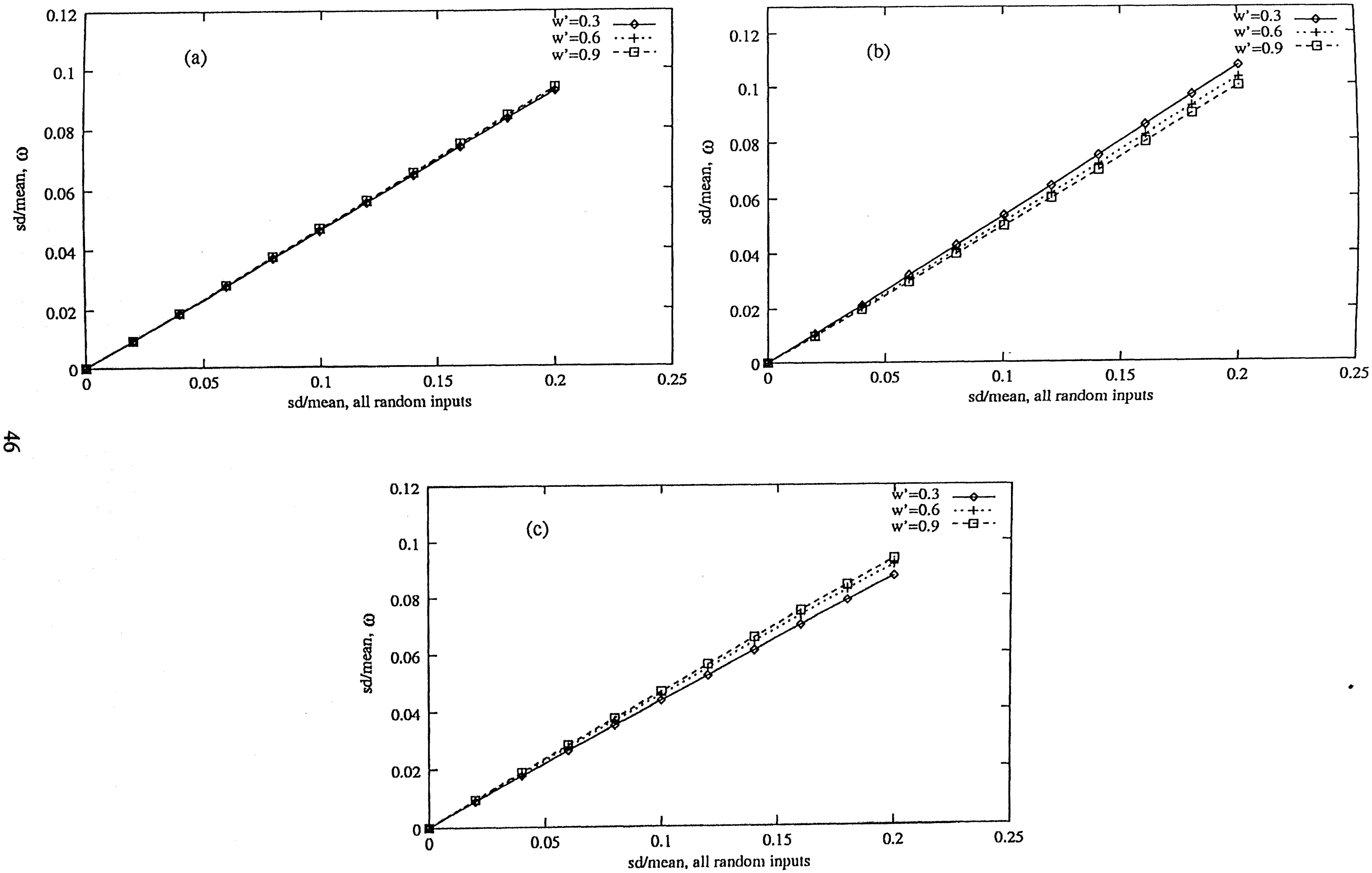


Figure 3.9- Influence of SD of all basic random inputs changing simultaneously on coefficient of variation of frequency of different cross-ply with b/h=100.
(a) $[0^0/90^0/90^0/0^0]$ laminate with a/b=1, (b) $[0^0/90^0/0^0/90^0]$ laminate with a/b=1, (c) $[0^0/90^0/90^0/0^0]$ laminate with a/b=2.0

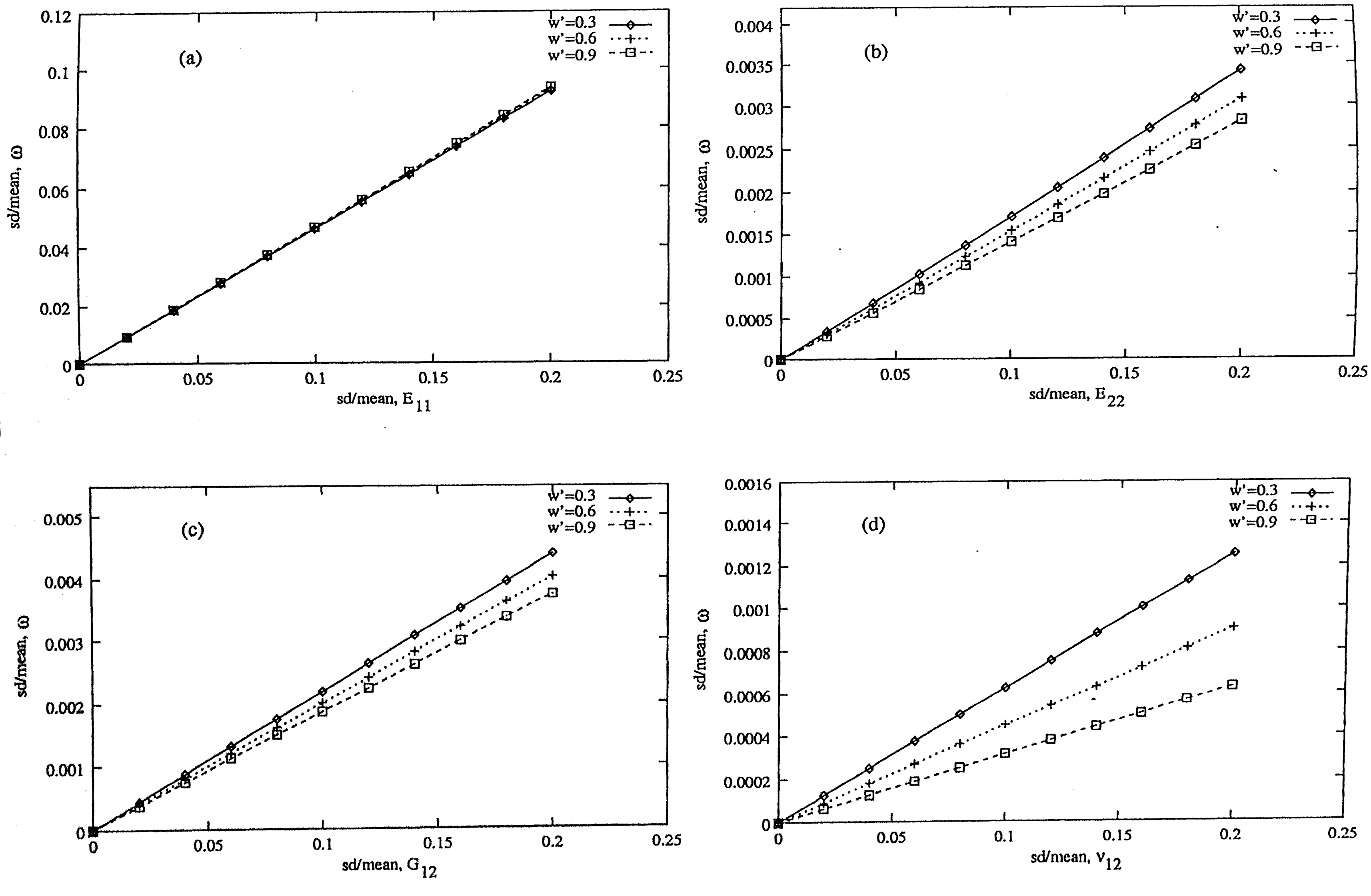


Figure 3.10- Influence of SD of basic material properties on coefficient of variation of frequency for $[0^0/90^0/90^0/0^0]$ laminate with a/b=1.0 and b/h=100. (a) only E_{11} varying, (b) only E_{22} varying, (c) only G_{12} varying, (d) only ν_{12} varying

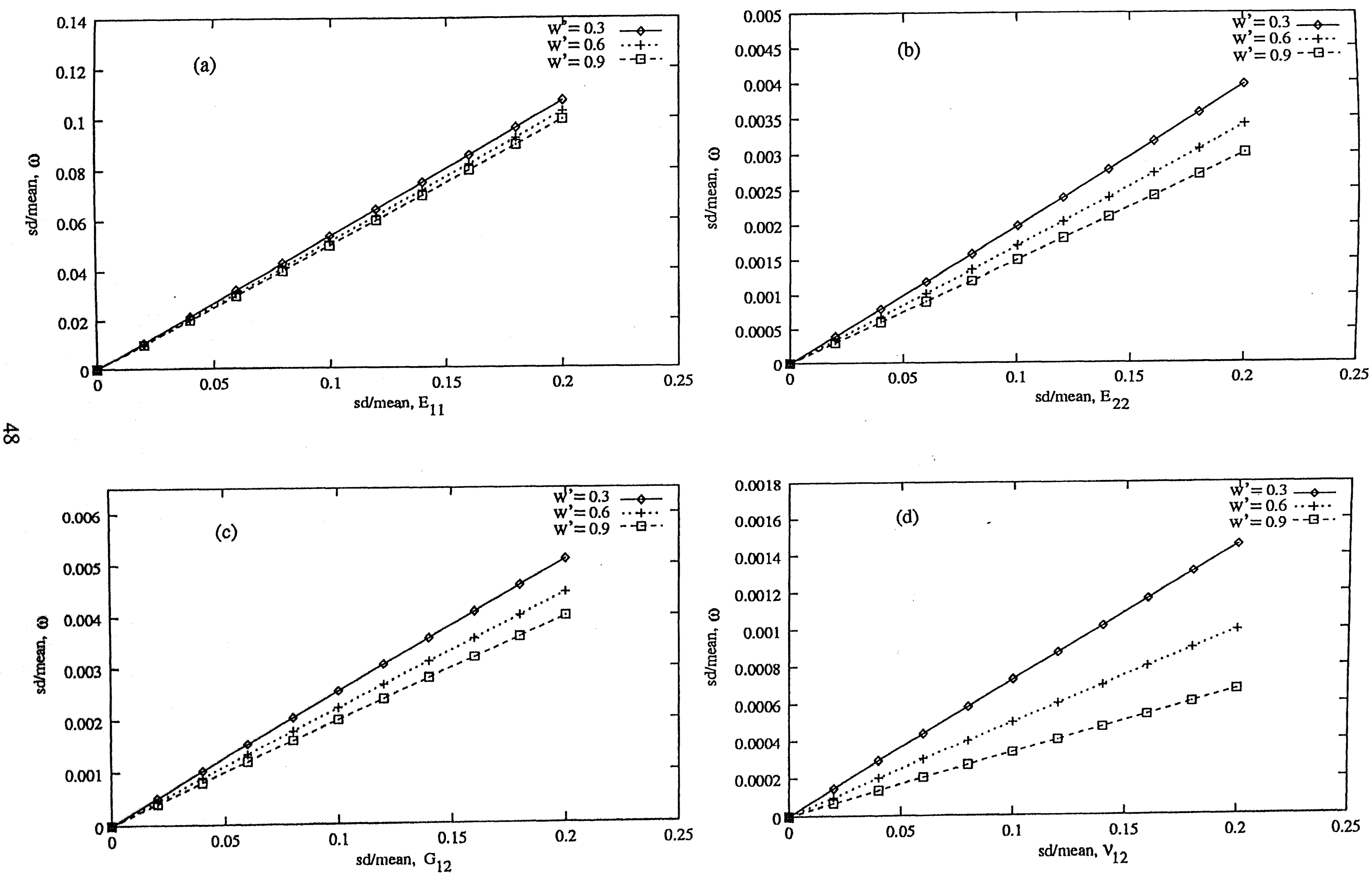


Figure 3.11- Influence of SD of basic material properties on coefficient of variation of frequency for [$0^0/90^0/0^0/90^0$] laminate with a/b=1.0 and b/h=100. (a) only E_{11} varying, (b) only E_{22} varying, (c) only G_{12} varying, (d) only ν_{12} varying

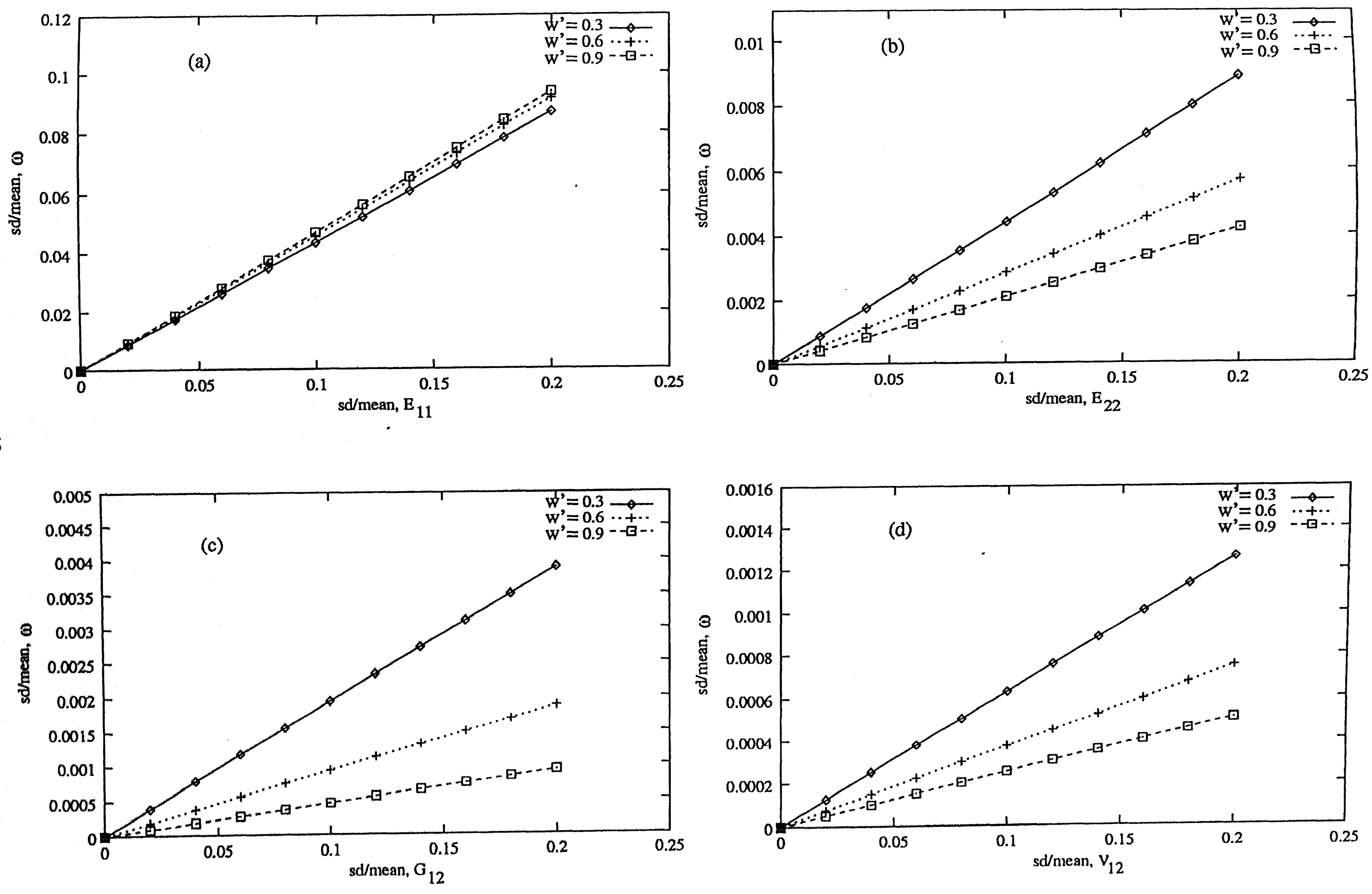


Figure 3.12- Influence of SD of basic material properties on coefficient of variation of frequency for $[0^0/90^0/90^0/0^0]$ laminate with a/b=2.0 and b/h=100. (a) only E_{11} varying, (b) only E_{22} varying, (c) only G_{12} varying, (d) only ν_{12} varying

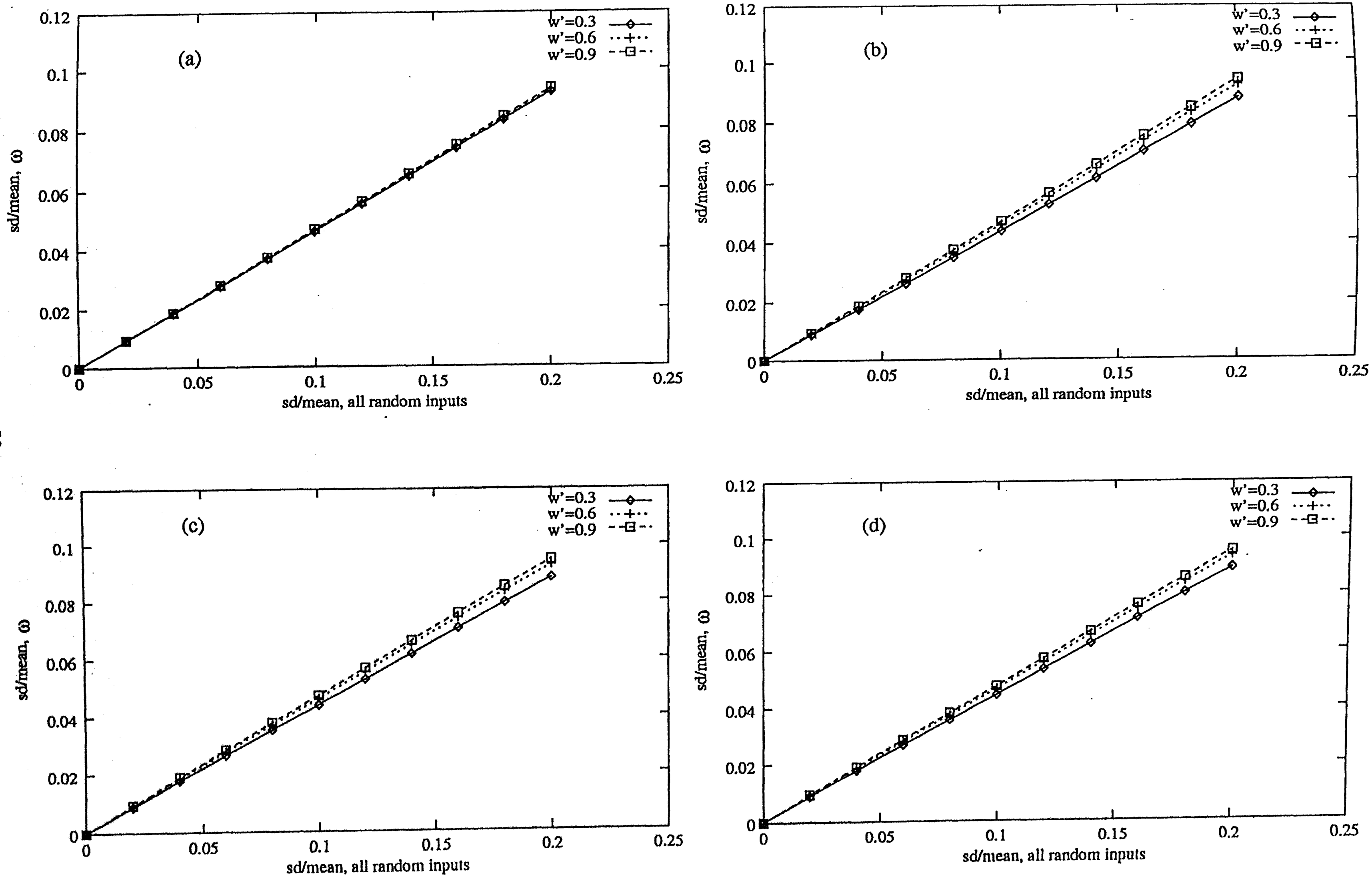


Figure 3.13- Influence of SD of all basic random inputs changing simultaneously on coefficient of variation of frequency for $[0^0/90^0/90^0/0^0]$ laminate having different aspect ratio of plate. (a) a/b=1, (b) a/b=2, (c) a/b=3 and (d) a/b=4

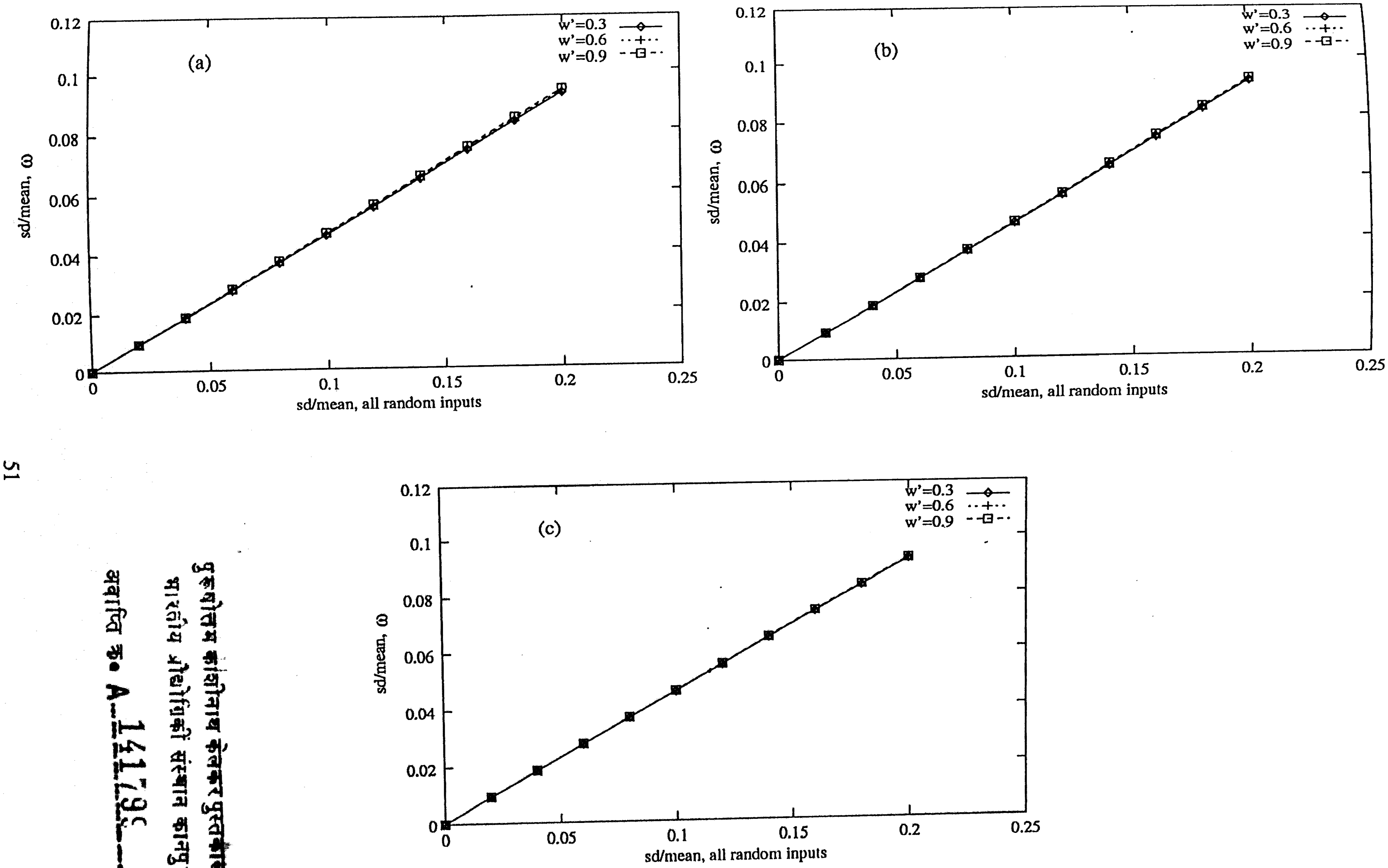


Figure 3.14- Influence of SD of all basic random inputs changing simultaneously on coefficient of variation of frequency for $[0^0/90^0/90^0/0^0]$ laminate having different plate thickness with a/b=1. (a) b/h=100, (b) b/h=50 and (c) b/h=33.33

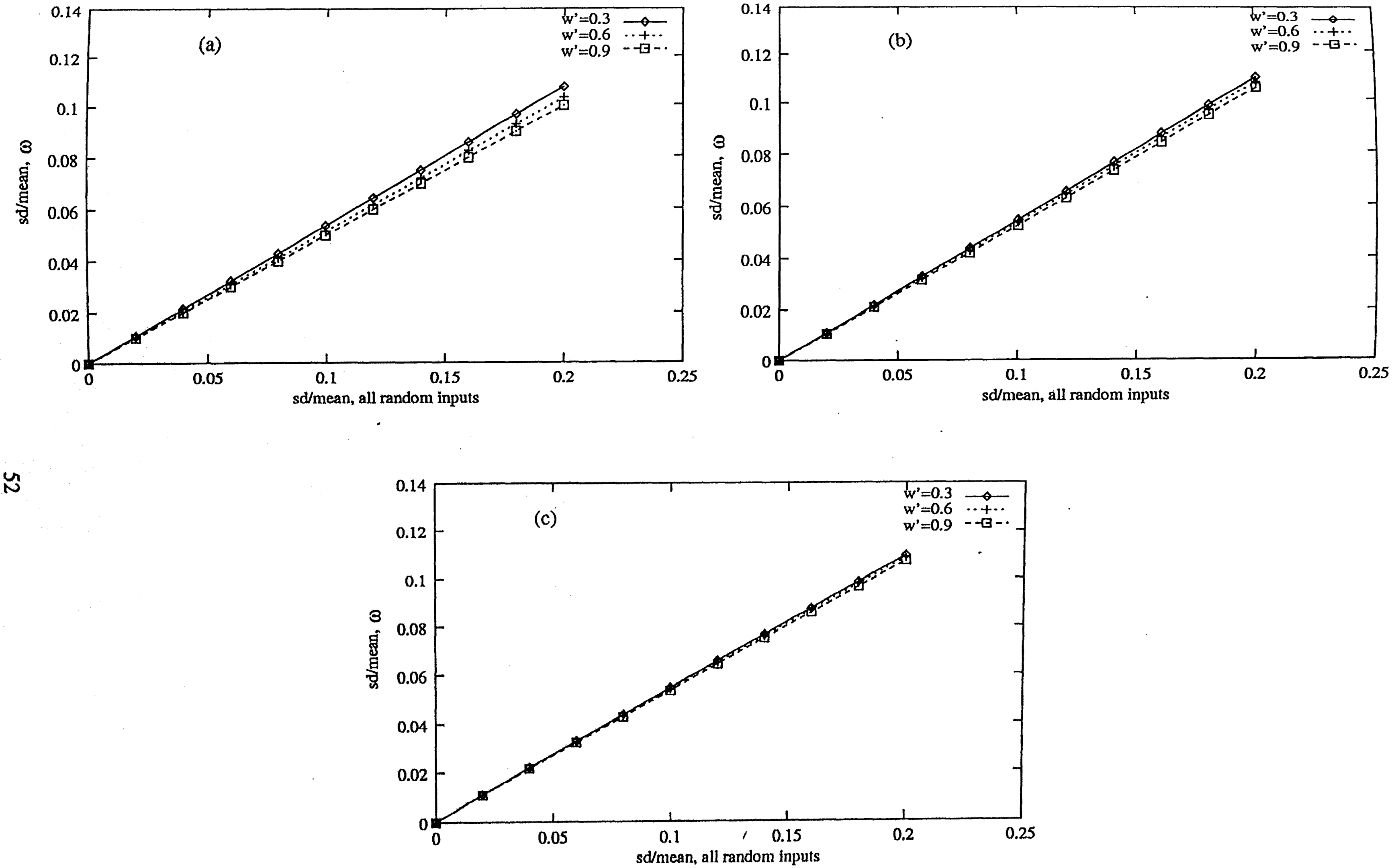
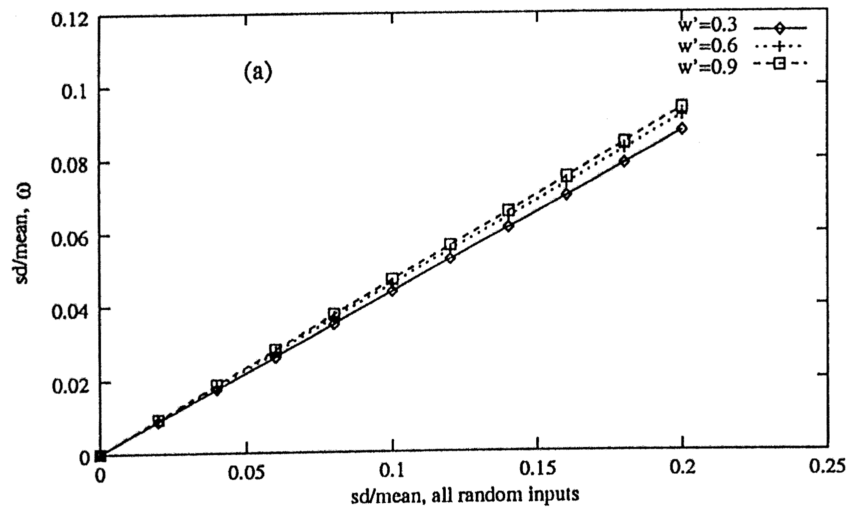
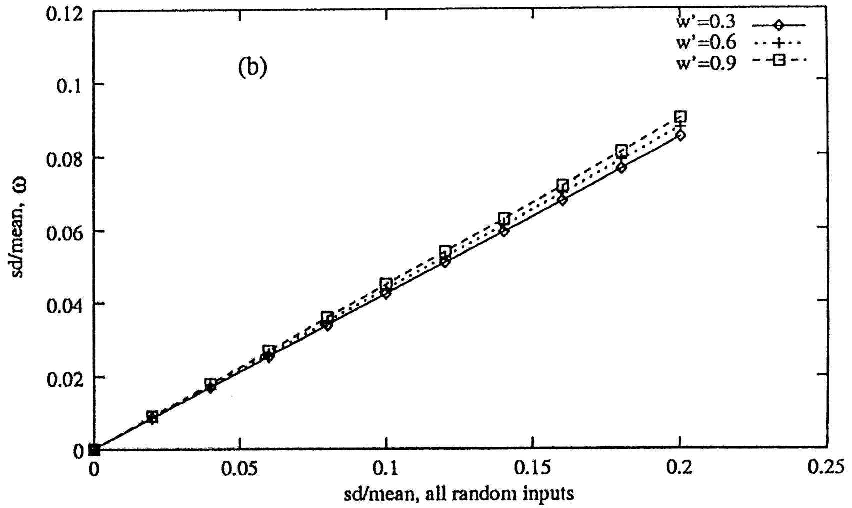


Figure 3.15- Influence of SD of all basic random inputs changing simultaneously on coefficient of variation of frequency for $[0^0/90^0/0^0/90^0]$ laminate having different plate thickness with a/b=1. (a) b/h=100, (b) b/h=50 and (c) b/h=33.33





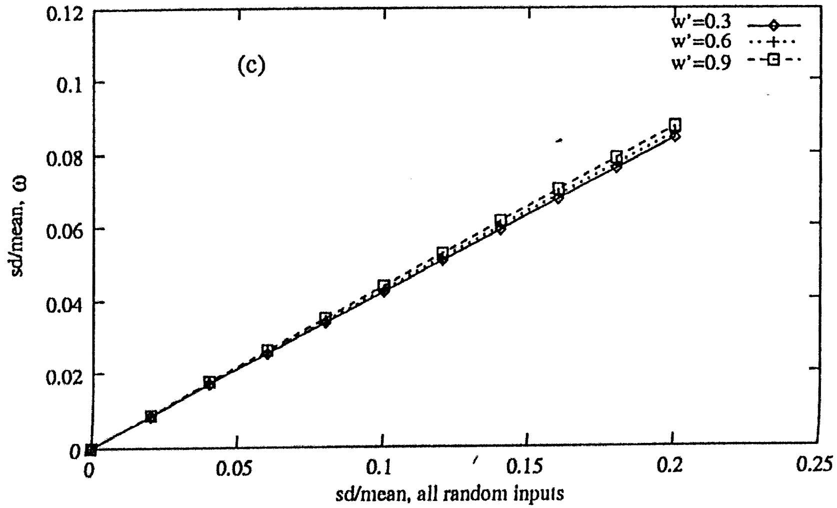


Figure 3.16- Influence of SD of all basic random inputs changing simultaneously on coefficient of variation of frequency for $[0^0/90^0/90^0/0^0]$ laminate having different plate thickness with a/b=2. (a) b/h=100, (b) b/h=50 and (c) b/h=33.33

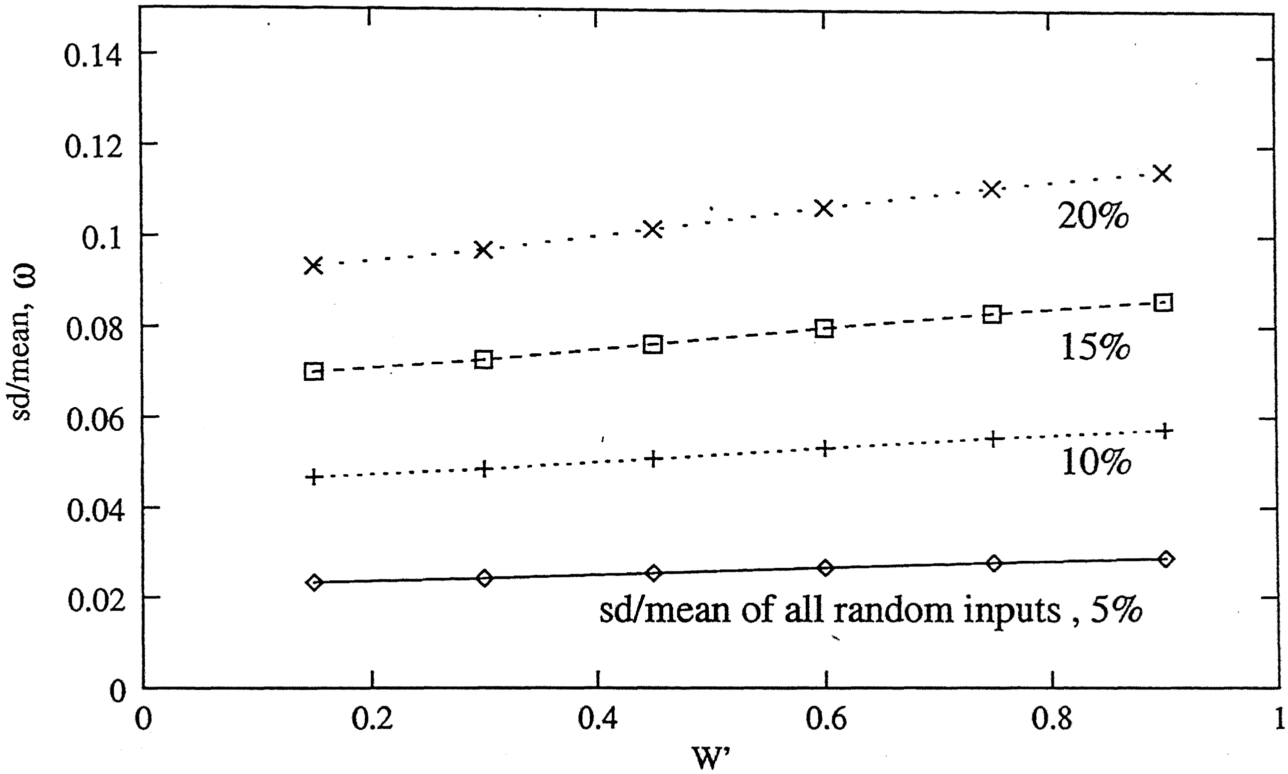


Figure 3.17- Influence of amplitude on coefficient of variation of frequency for $[0^0/90^0/90^0/0^0]$ laminate with all material properties changing simultaneously

The reference uses direct numerical integration method for the analysis of mean frequency whereas the present approach gives an exact solution. A reasonably good agreement between the two is observed.

3.3.2 Second order amplitude statistics

The material used for the graphite/epoxy composite plate is the same as employed for generating Table 3.7. All the four material properties and excitation are considered as random for the analysis.

3.3.2.1 Mean amplitude

Table 3.8 presents the non-dimensionalised mean nonlinear frequency for different plate thickness and amplitudes with simply supported edges. Influence of the scattering in the material properties on the mean amplitude has been obtained by allowing the coefficient of variation to change from 0 to 20 % for laminated cross ply plates. The plates have aspect ratios $b/a=1$ and thickness ratios b/h = 100, 50 and 33.33 with stacking sequences of [0°/90°/90°/0°] and [0°/90°/0°/90°]. The mean amplitude decreases with increase in the plate thickness and increases with the oscillation amplitude. The anti-symmetric laminate has higher mean amplitude compared to the symmetric laminate.

3.3.2.2 Variance of amplitude

The variations of non-dimensionalised amplitude with dispersion in all the basic inputs changing simultaneously are presented in Figures 3.18 and 3.19. Two cases square symmetric and anti-symmetric four layered cross ply with b/h=100 have been examined. It is seen that as excitation frequency increases the amplitude sensitivity increases for all

cases. The sensitivity of the square anti-symmetric cross ply is greater than the symmetric case. Here, the graphs are presented for excitation frequencies ω_1=31.62, ω_2=47.43 and ω_3=63.34 rad/sec.

Figures 3.20 (a) - (d) and 3.21 (a) - (d) present the influence of only one basic material property random at a time on displacement response dispersion for square symmetric and antisymmetric cross ply with b/h=100. The amplitude variations are most affected by change in E_{11} and least affected by dispersion in ν_{12}. It is also seen that sensitivity of the oscillation amplitude to plate frequency increases with increased variations in E_{11}, E_{22}, G_{12} and ν_{12}.

Figures 3.22 and 3.23 show amplitude sensitivity to dispersion in excitation for square symmetric and antisymmetric cross-ply respectively for b/h=100. It is seen that variation of excitation has a dominant effect on the deflection scattering as compared to E_{22}, G_{12} and ν_{12}.

Influence of frequency on coefficient of variation of amplitude with a constant SD/mean for all basic inputs changing simultaneously for square symmetric cross-ply with b/h=100 is shown in Figure 3.24. It is found that the amplitude scatter increases nonlinearly with increase in frequency and also increases with increase in variation of basic inputs.

The interrelation between (w_{max}/h) on coefficient of variation of amplitude with a constant SD/mean for all basic inputs changing simultaneously for square symmetric cross-ply with b/h=100 is shown in Figure 3.25. This also shows that the amplitude scatter increases nonlinearly with increase in amplitude and also increases with increase in variation of basic inputs.

Table 3.7: Comparison of the nondimensionalised mean nonlinear frequency (ω_{nl}/ω_{l}) for [0°/90°] laminate with different amplitude and aspect ration a/b=1

w_{max}/h	Present work	Ref[19]
+0.4	0.8943	0.8898
-0.4	1.3408	1.3395
+0.6	1.1595	1.1486
-0.6	1.4167	1.4109
+1.0	1.6428	1.6116
-1.0	1.7581	1.7334

Table 3.8: Nondimensionalised mean nonlinear frequency (ω_{nl}/ω_{l}) for different square plates with different amplitudes

w_{max}/h	(ω_{nl}/ω_{l})	
	Stacking sequence [0°/90°/90°/0°]	Stacking sequence [0°/90°/0°/90°]
+0.4	0.8984	0.9311
-0.4	1.3435	1.3654
+0.6	1.1667	1.2233
-0.6	1.4224	1.4689
+1.0	1.6568	1.7676
-1.0	1.7718	1.8749

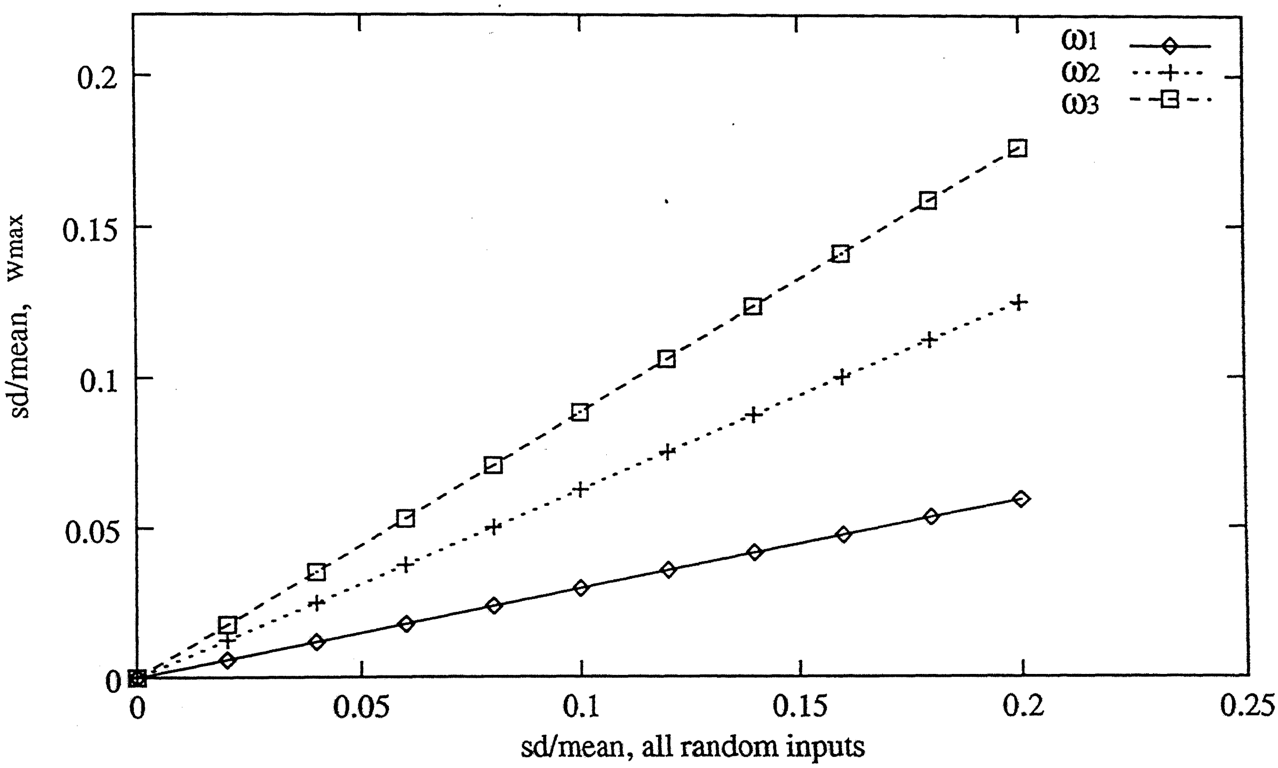


Figure 3.18- Influence of SD of all basic random inputs changing simultaneously on coefficient of variation of amplitude for [$0^0/90^0/90^0/0^0$] laminate with a/b=1 and b/h=100

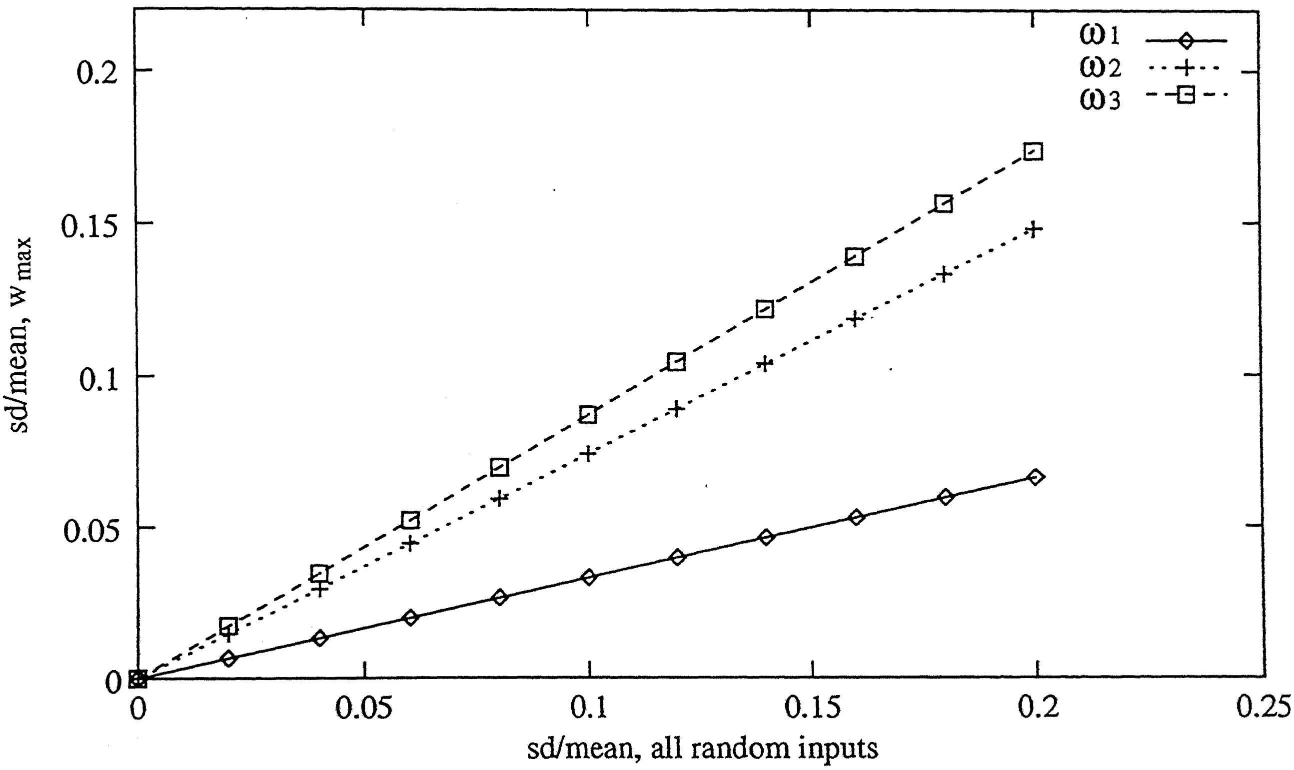


Figure 3.19- Influence of SD of all basic random inputs changing simultaneously on coefficient of variation of amplitude for $[0^0/90^0/0^0/90^0]$ laminate with a/b=1 and b/h=100

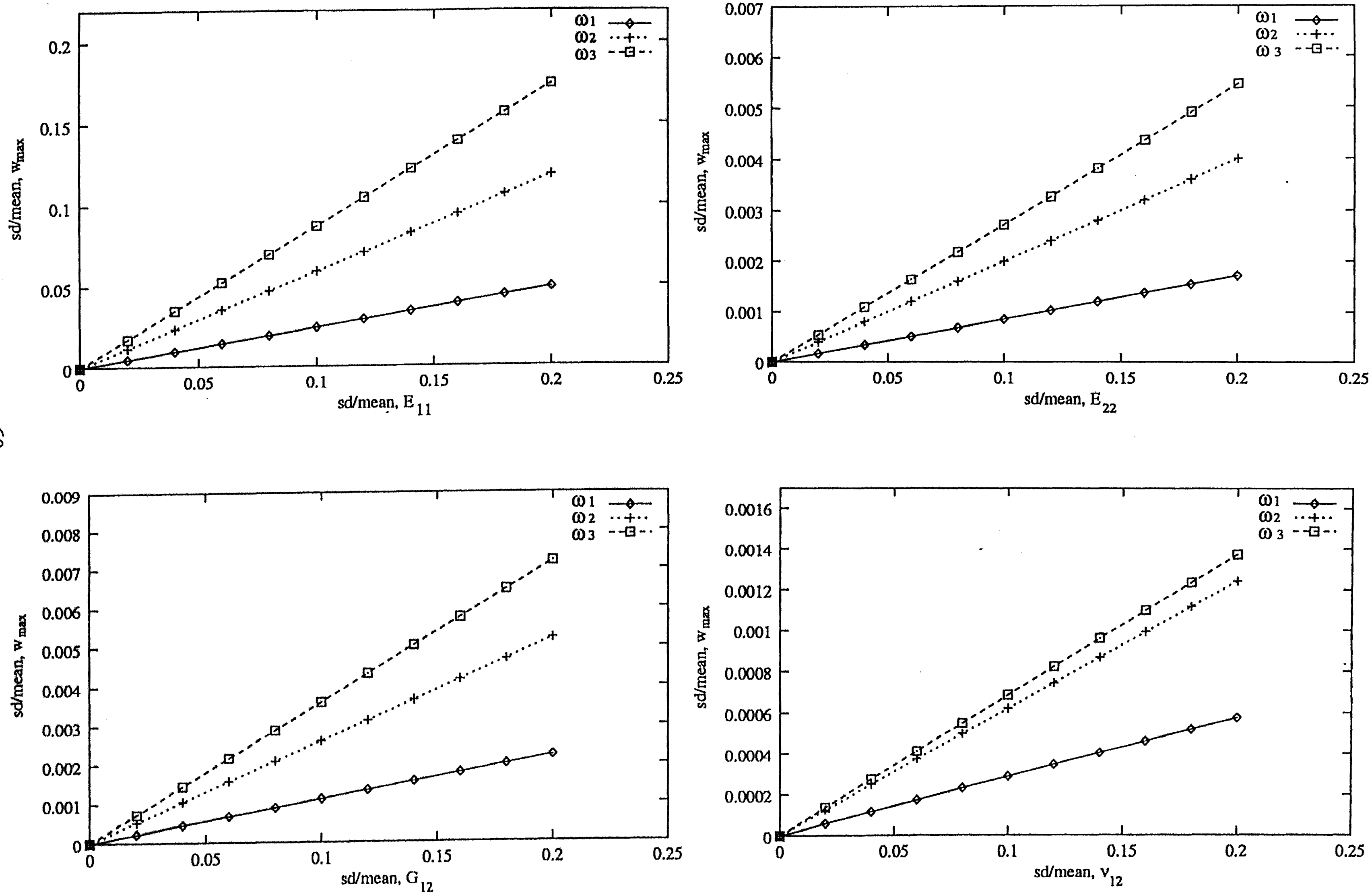


Figure 3.20- Influence of SD of basic material properties on coefficient of variation of amplitude for $[0^0/90^0/0^0/90^0]$ laminate with a/b=1.0 and b/h=100. (a) only E_{11} varying, (b) only E_{22} varying, (c) only G_{12} varying, (d) only ν_{12} varying

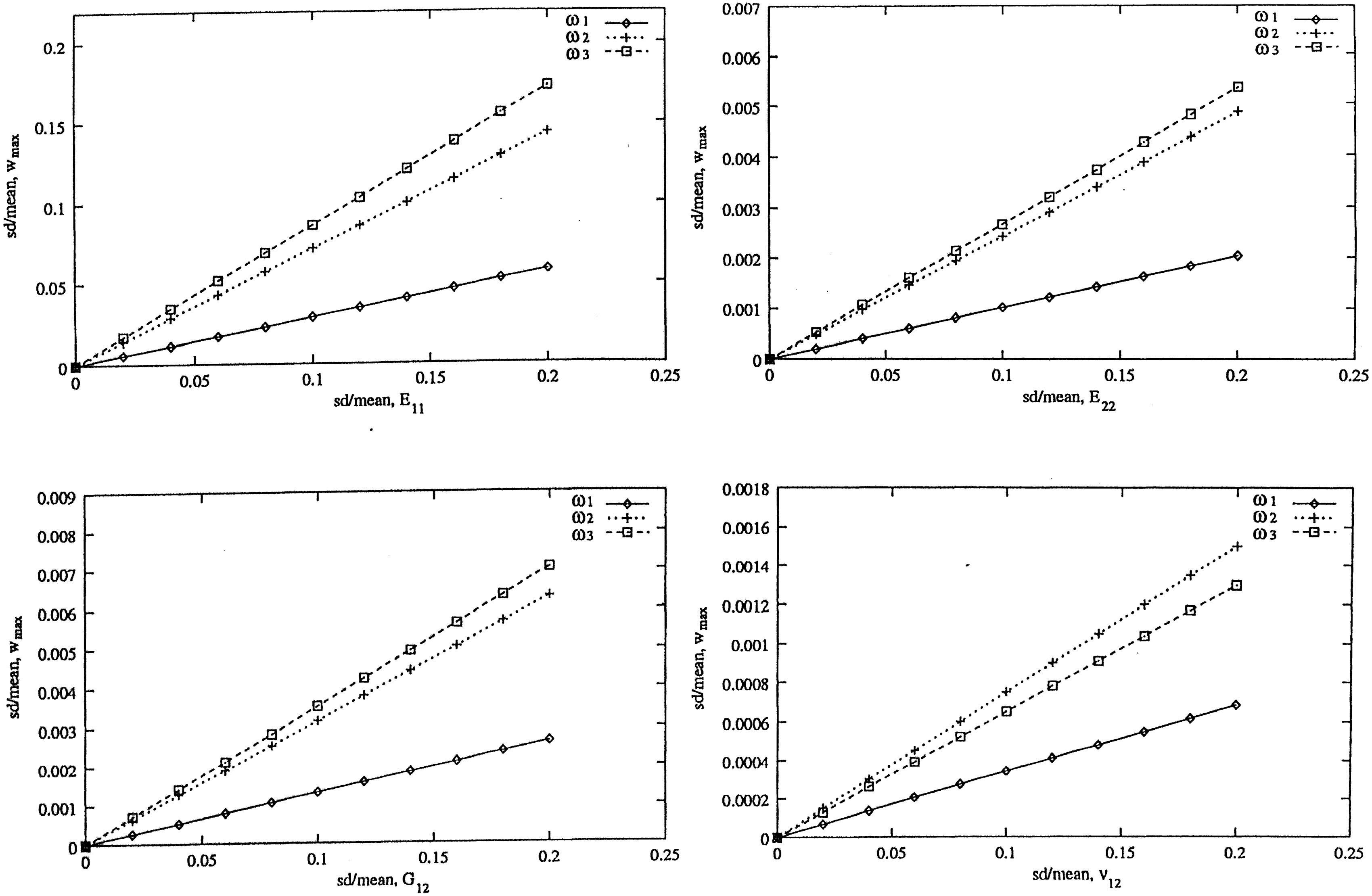


Figure 3.21- Influence of SD of basic material properties on coefficient of variation of amplitude for $[0^0/90^0/90^0/0^0]$ laminate with *a/b*=2.0 and *b/h*=100. (a) only E_{11} varying, (b) only E_{22} varying, (c) only G_{12} varying, (d) only ν_{12} varying

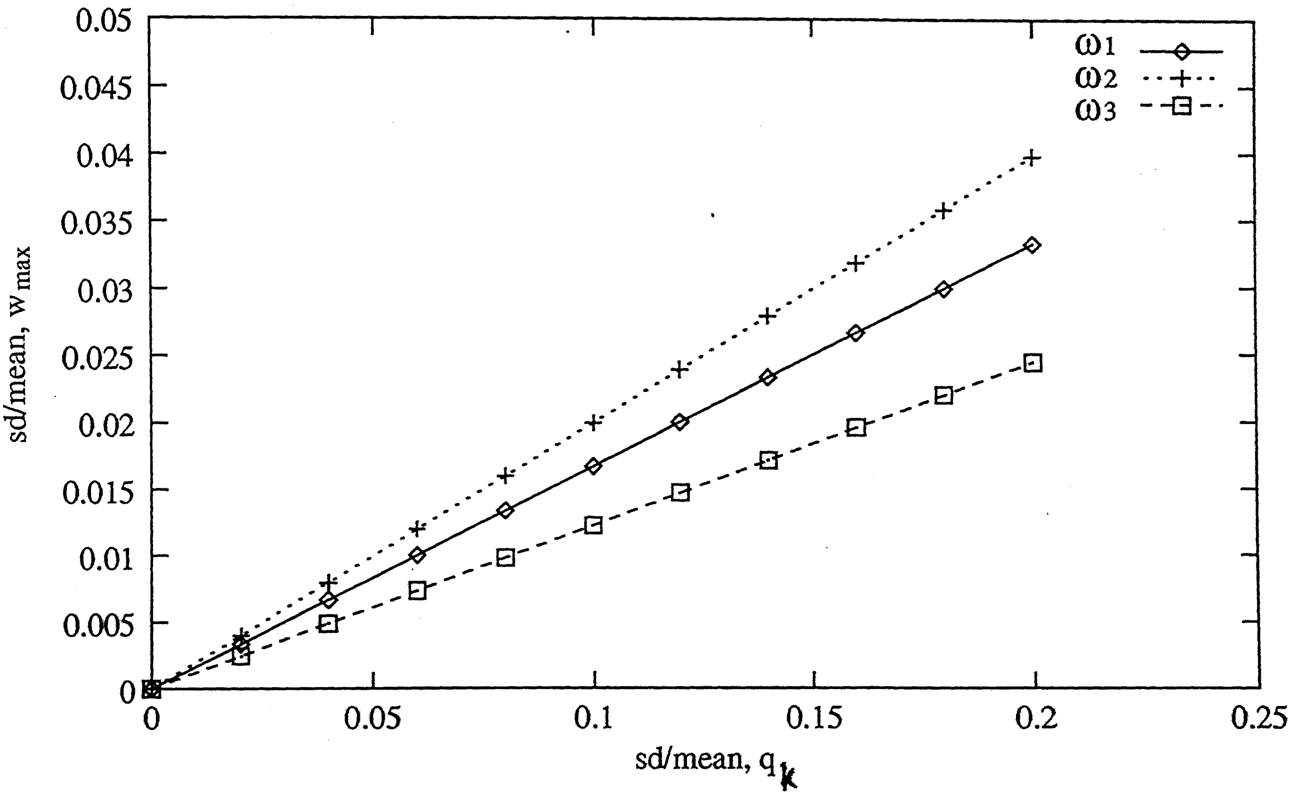


Figure 3.22- Influence of SD of excitation 'q_k' on coefficient of variation of amplitude for [$0^0/90^0/90^0/0^0$] laminate with a/b=1.0and b/h=100

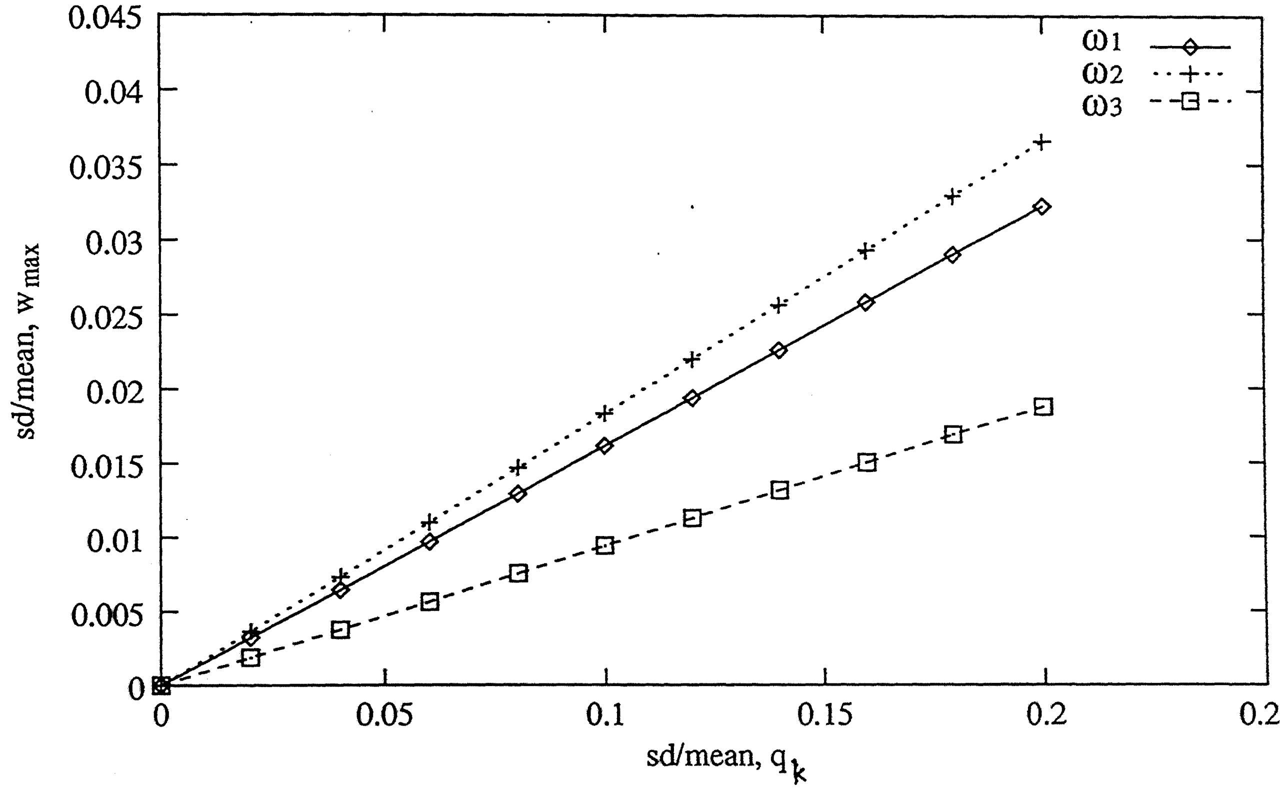


Figure 3.23- Influence of SD of excitation 'q_k' on coefficient of variation of amplitude for [0^0/90^0/90^0/0^0] laminate with *a/b*=1.0and *b/h*=100

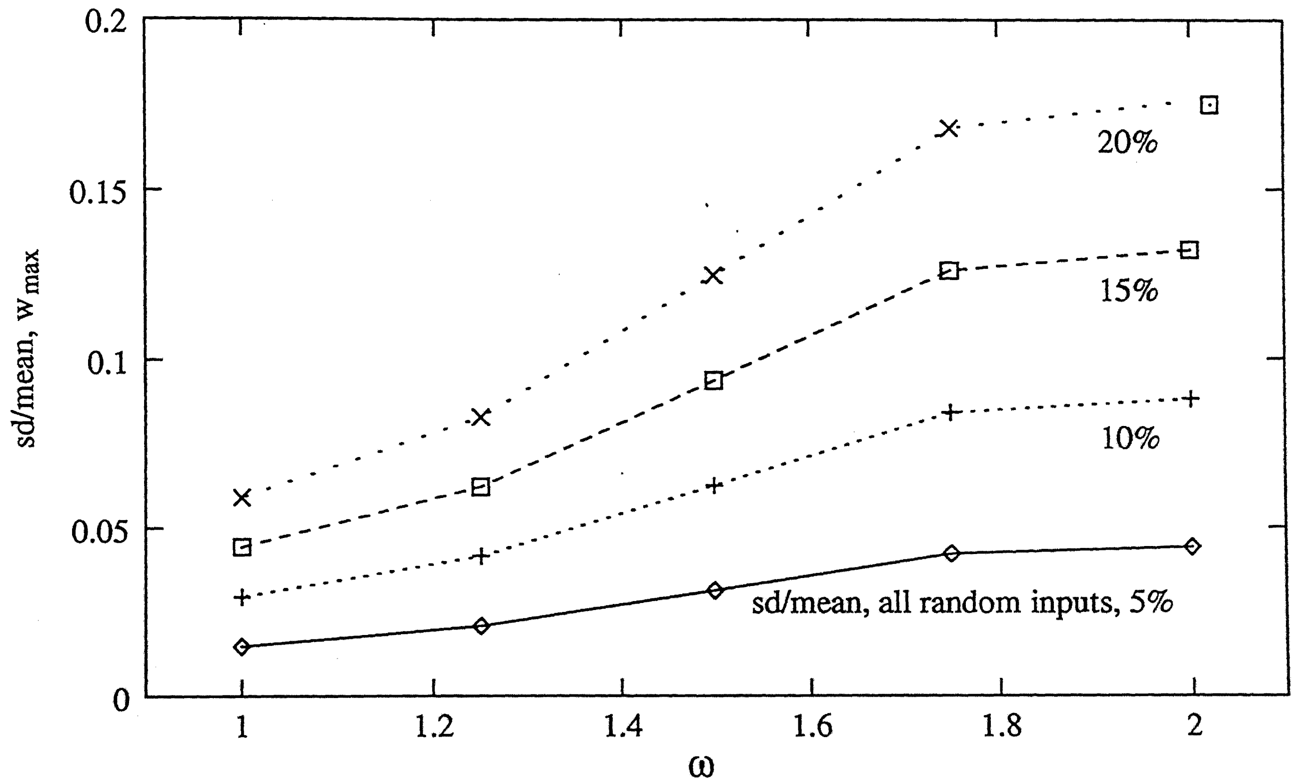


Figure 3.24- Influence of frequency on coefficient of variation of amplitude for $[0^0/90^0/90^0/0^0]$ laminate for all random inputs changing simultaneously

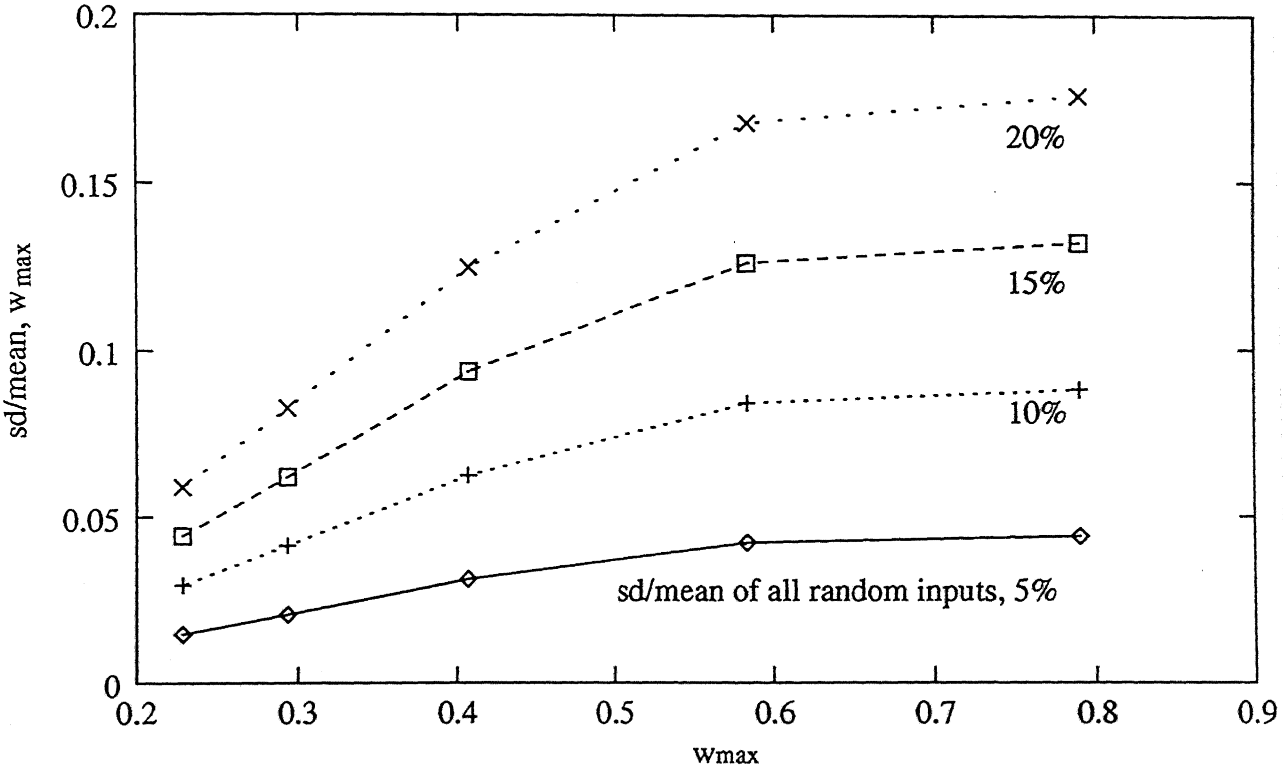


Figure 3.25- Influence of amplitude on coefficient of variation of amplitude for $[0^0/90^0/90^0/0^0]$ laminate with all random inputs changing simultaneously

CHAPTER 4

CONCLUSIONS

An approach has been presented in *Chapter* 2 to obtain the second order statistics for static deflection, free vibration and forced vibration response of laminated composite plates with random material properties under random loading for nonlinear strain-displacement relations. The system equation for the plate has been solved using termwise series integration and a first order perturbation technique is used to obtain the second order statistics.

Results obtained using the present approach have been discussed in *Chapter* 3 for two stacking sequences of cross-ply laminates. The following conclusions are noted from the results for the graphite-epoxy laminated plate having all edges simply supported with movable edges.

Static deflection

(i) The dispersions in the static deflection show linear variation with SD of the material properties and the external loading in the range studied. One of the main reason for this may be the FOPT employed in developing the analysis.

(ii) The influence of SD of static deflection shows different sensitivity to different input variables. The sensitivity changes with the laminate construction, a/h ratios and the material. However, any recognizable pattern is not apparent in all cases.

(iii) Increase in thickness ratio of the plate results in increase in deflection scatter whereas in case of linear analysis plate thickness ratio has no effect.

(iv) Increase in the aspect ratio of the plate results in decrease in deflection scatter.

(vi) Variation of E_{11} and load Q_0 has dominant effect on the scattering of deflection as compared to E_{22}, G_{12} and ν_{12}.

(vii) Variation in the external load causes larger variations in the deflection response compared to the effect of material properties.

(viii) Increase in mean value of load results in decrease in deflection scatter whereas it is predicted to be constant by linear analysis.

Free vibration

(i) The dispersions in the frequency response show linear variation with SD of the material properties in the range studied whereas nonlinear variation in frequency has been seen with variation in the oscillation amplitude.

(ii) Increase in thickness ratio and oscillation amplitude of the plate results in increase in frequency scatter with all material properties changing simultaneously. Linear analysis shows no dependence of frequency on plate thickness ratio and oscillation amplitude.

(iii) Variation of E_{11} has dominant effect on the scattering of frequency as compared to E_{22}, G_{12} and ν_{12}.

Forced vibration

(i) The dispersions in the response amplitude show linear variation with SD of the material properties in the range studied whereas nonlinear variation in amplitude has been seen with variation in the oscillation amplitude and frequency.

(ii) Variation of E_{11} has dominant effect on the scattering of amplitude as compared to E_{22}, G_{12} and ν_{12} and the excitation Q_0.

(iii) Variation in the external load causes larger variations in the deflection response compared to the effect of material properties. This is also observed in the static case.

SUGGESTIONS FOR FUTURE WORK

The present study gives a first attempt at nonlinear analysis for laminated composite plates with random material properties subjected to random loading. The following extensions can be applied to the present problem.

(i) The problem can be solved by using higher order perturbation technique by taking higher order terms in Taylor series expansion of material properties and loading to get higher dispersion.

(ii) Effect of correlation between the material properties as well as between material properties and loading on the response should be studied.

(iii) Material properties can be assumed to vary with respect to space and time to incorporate the aging effect as well as point to point variations in composites.

(iv) Random loading can be assumed to vary with respect to space also.

(v) Geometric parameters and boundary condition can also be modeled as random.

(vi) Effect of FSDT and HSDT on the response can be examined to extend the formulation to thick plates.

(vii) Effect of hygrothermal behaviour on the response can also be examined.

(viii) The formulation can be extended to curved panels.

REFERENCES

[1] Nigam, N.C. and Narayanan, S., *Application of random vibration*, Narosa, New Delhi, 1994.

[2] Ibrahim, R.A., 'Structural dynamics with parameter uncertainties', Trans. ASME, *Applied mechanics review* **40**(3) 1987, 309-328.

[3] Nakagiri, S., Takabatake, H. and Tani, S., 'Uncertain eigen value analysis of composite laminated plates by the sfem', Trans. ASME *Journal of Engineering for Industry* **109**, 1987, 9-12.

[4] Fukunaga, H., Chow, T.W. and Fukuda, H., 'Probabilistic strength analysis of inter laminated hybrid composite', *Composite Science Technology* **35**, 1989, 331-45.

[5] Raj, B.N., Iyengar, N.G.R. and Yadav, D., 'Response of composite plates with random material properties using Monte Carlo Simulation', *Advanced composite Material* **7**(3), 1998, 219-37.

[6] Leissa, A.W. and Martin, A.F., 'Vibration and buckling of rectangular composite plates with variable fiber spacing', *Composite Structures* **14**, 1990, 339-357.

[7] Shinozuna, M. and Astill, C.J., 'Random eigenvalue problems in structure analysis', *American Institute of Aeronautics and Astronautics Journal* **10**(4), 1972, 456-462.

[8] Singh, G., Venkateswara, R. and Iyengar, N.G.R., 'Some observation on the large deflection of rectangular antisymmetric cross-ply plates', *Composite Structure* **18**, 1991, 77-91.

[9] Salim, S., Yadav, D. and Iyengar, N.G.R., 'Deflection of composite plates with random material characteristics', Proceedings of symposium on Recent Advances in

Aerospace Science and Engineering Conference, Bangalore, New Delhi **1**, 1992, 236-239.

[10] Salim, S., Yadav, D. and Iyengar, N.G.R., 'Analysis of composite plates with random material characteristics', *Mechanics Research Communication* **20**(5), 1993, 405-414.

[11] Salim, S., 'Analysis of composite plates with randomness in material properties', *PhD thesis* , Dept of Aerospace Engg, IIT Kanpur, 1995.

[12] Salim, S., Yadav, D. and Iyengar, N.G.R., 'Buckling of laminated plates with random material characteristic', *Applied Composite Material* **5**, 1998, 1-9.

[13] Vaicatis, R., 'Free vibrations of beam with random characteristics', *Journal of Sound and Vibration* **45**(1), 1975, 13-21.

[14] Singh, G., Venkateswara, R. and Iyengar, N.G.R., 'Large amplitude free vibration of simply supported antisymmetric cross-ply cross-ply plates', *American Institute of Aeronautics and Astronautics Journal* **29**(5), 1990, 784-790.

[15] Lin, Y.K., 'Response of a nonlinear flat panel to periodic and randomly varying loading', *Journal of Aerospace Science* **29**, 1962, 1029-34.

[16] Srinivasan, A.V., 'Nonlinear vibration of beams and plates', *International Journal of Nonlinear Mechanics* **1**, 1966, 179-91.

[17] Kung, G.C. and Pao, Y.H., 'Nonlinear flexural vibrations of a clamped circular plate', *Journal of Applied Mechanics Trans ASME* E-39, 1972, 1050-1054.

[18] Yamaki, N., Otomo, K. and Chiba, M., 'Nonlinear vibration of a clamped circular plates with initial deflection and initial edge displacement', in Part –I *Journal of Sound and Vibration* **79**, 1981, 23-42.

[19] Singh, G., Venkateswara, R. and Iyengar, N.G.R., 'Nonlinear forced vibrations of antisymmetric rectangular cross-ply plates', *Composite Structure* **18**, 1991, 77-91.

[20] Chen, P.C. and Soroka, W.W., 'Multi-degree dynamics response of a system with statistical properties' *Journal of Sound and Vibration* **37**(4), 1974, 547-556.

[21] Yadav, D. and Verma, N., 'Buckling of composite circular cylindrical shells with random material properties', *Composite Structure* **37**, 1997, 385-391.

[22] Yadav, D. and Verma, N., 'Free vibration of composite circular cylindrical shell with random material properties Part I: General theory' *Composite Structure* **41**, 1998, 331-338.

[23] Gorman, D.J., 'Free vibration analysis of rectangular plates with non-uniform lateral elastic edge support', *Applied Mechanics Reviews* **60,** 1993, 998-1003.

[24] Singh, B.N., Yadav, D. and Iyengar, N.G.R., 'Initial buckling of composite cylindrical panels with random material properties', *Composite Structures* **53**, 2001, 55-64.

[25] Singh, B.N., Yadav, D. and Iyengar, N.G.R., 'Natural frequencies of composite plates with random material properties using higher-order shear deformation theory', *Mechanical Sciences* **43**, 2001, 2193-2214.

[26] Jones, R.M., *Mechanics of composite Materials*, McGrow-Hill, New York, 1975.

[27] Chia, C. Y., *Nonlinear analysis of plates*, McGrow-Hill, New York, 1980.

[28] Crandall, S. H., *Random Vibration*, John Wiley and Sons, Inc., New York, 1958.

[29] Vinson, J. R. and Sierakowski, R., *The behaviour of structures of composite material*, Martinus Nijhoff Publishers, Dordrecht, 1986.

[30] Whitney, J. M., Structural analysis of laminated anisotropic plates, Technomic, Lancaster, 1987.

[31] Turvey, G.J. and Marshall, I. H., Buckling and postbuckling of composite plates, Chapman and Hall, London, 1995.